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Basic and equivariant cohomology in balanced topological field theory

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Abstract

We present a detailed algebraic study of the $N = 2$ cohomological set-up describing the balanced topological field theory of Dijkgraaf and Moore. We emphasize the role of $N = 2$ topological supersymmetry and $\mathfrak{sl}(2, \mathbb{R})$ internal symmetry by a systematic use of superfield techniques and of an $\mathfrak{sl}(2, \mathbb{R})$ covariant formalism. We provide a definition of $N = 2$ basic and equivariant cohomology, generalizing Dijkgraaf's and Moore's, and of $N = 2$ connection. For a general manifold with a group action, we show that: (i) the $N = 2$ basic cohomology is isomorphic to the tensor product of the ordinary $N = 1$ basic cohomology and a universal $\mathfrak{sl}(2, \mathbb{R})$ group theoretic factor; (ii) the affine spaces of $N = 2$ and $N = 1$ connections are isomorphic.

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1. Introduction

Topological quantum field theories are complicated, often fully interacting, local renormalizable field theories, yet they can be solved exactly and the solution is highly nontrivial. Expectation values of topological observables provide topological invariants of the manifolds on which the fields propagate. These invariants are independent from the couplings and to a large extent from the interactions between the fields. At the same time, topological field theories are often topological sectors of ordinary field theories. In this way, they are convenient testing grounds for subtle nonperturbative field theoretic phenomena. See, e.g., Refs. [1–3] for an updated comprehensive review on the subject and complete referencing.

$N = 1$ cohomological topological field theories have been the object of intense and exhaustive study. They can be understood in the framework of equivariant cohomology

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of infinite dimensional vector bundles [4–9] and realized as Mathai–Quillen integral representations of Euler classes [10–13]. The resulting formalism is elegant and general and covers the important case where the quotient by the action of a gauge symmetry group is required. Each of these models describes the differential topology of a certain moduli space, depending on the model considered: the field theoretic correlation functions of topological observables correspond to intersection numbers on the moduli space.

$N = 2$ cohomological topological field theories were discovered quite early [14–17], but they did not arouse much interest until recently when it became clear that they might provide important clues in the analysis of S duality in supersymmetric Yang–Mills theory and in the study of the world volume theories of D-branes in string theory.

In Ref. [18], Vafa and Witten performed an exact strong coupling test of S duality of $N = 4$ supersymmetric 4-dimensional Yang–Mills theory by studying a topological twist of the model yielding an $N = 2$ cohomological field theory. They showed that the partition function is $Z(\tau) = \sum_k a_k \exp(2\pi i \tau k)$, where a_k is the Euler characteristic of the moduli space of k instantons, and tested S duality by analyzing the modularity properties of $Z(\tau)$. Their work, inspired by the original work of Yamron [14], was soon developed and refined in a series of papers [19–24]. In Ref. [25], Bershadsky et al. showed that the three $N = 2$ cohomological topological field theories obtained by the nontopological twistings of $N = 4$ supersymmetric 4-dimensional Yang–Mills theory arose from curved 3-branes embedded in Calabi–Yau manifolds and manifolds with exceptional holonomy groups. Their analysis was continued and further developed in Refs. [21,26–28], where the connection with higher dimensional instantons was elucidated. In Ref. [29], Park constructed a family of Yang–Mills instantons from D-instantons in topological twisted $N = 4$ supersymmetric 4-dimensional Yang–Mills theory. In Ref. [30], Hofman and Park worked out a 2-dimensional $N = 2$ cohomological topological field theory as a candidate for covariant second quantized RNS superstrings, which they conjectured to be a formulation of M theory.

All the endeavors mentioned above, and many other related ones, which we cannot mention for lack of space, show that $N = 2$ cohomological topological field theories are relevant in a variety of physical and mathematical issues. In spite of that, the body of literature devoted to the study of the geometry of such models is comparatively small. In Ref. [17], Blau and Thompson worked out a Riemannian formulation of $N = 2$ topological gauge theory using $N = 2$ topological superfield techniques. In Ref. [31], Dijkgraaf and Moore showed that all known $N = 2$ topological models were examples of "balanced topological field theories" and developed a cohomological framework suitable for their study. In Ref. [21], Blau and Thompson proved the equivalence of their earlier formulation and Dijkgraaf's and Moore's. These studies show that the partition function of every $N = 2$ topological model calculates the Euler characteristic of some moduli space of vanishing virtual dimension. They also indicate that the appropriate cohomological scheme is provided by $N = 2$ basic or equivariant cohomology. The present paper aims at a systematic study of the latter developing the ideas of [31].

In general, a cohomological topological field theory is characterized by a symmetry Lie algebra $\mathfrak{g}$, a graded algebra of fields f and a set of graded derivations on f generating a Lie algebra $\mathfrak{t}$. In turn, the topological algebra $\mathfrak{t}$ provides the algebraic and geometric framework for the definition of the topological observables [1].

As is well known, in $N=1$ cohomological topological field theory, $\mathfrak{t}$ is generated by four derivations $k, d, j(\xi), l(\xi), \xi \in \mathfrak{g}$, of degrees $0, 1, -1, 0$, respectively, obeying the graded commutation relations given by Eqs. (29)–(33) below. k is the ghost number operator. d is the nilpotent topological charge. $j(\xi), l(\xi)$ describe the action of the symmetry Lie algebra $\mathfrak{g}$ on fields. The elements $\alpha \in \mathfrak{f}$ are classified into the eigenspaces $\mathfrak{f}^p$, $p \in \mathbb{Z}$, of k. The $N=1$ basic degree p cohomology of $\mathfrak{f}$ is defined by

$$j(\xi)\alpha = 0, \quad l(\xi)\alpha = 0, \quad \xi \in \mathfrak{g}, \quad d\alpha = 0, \tag{1}$$

$$\alpha \equiv \alpha + d\beta, \quad \beta \in \mathfrak{f}^{p-1}, \quad j(\xi)\beta = 0, \quad l(\xi)\beta = 0, \quad \xi \in \mathfrak{g}, \tag{2}$$

with $\alpha \in \mathfrak{f}^p$.

The $N=1$ Weil algebra $\mathfrak{w}$, an essential element of the definition of the $N=1$ equivariant cohomology of $\mathfrak{f}$, is generated by two $\mathfrak{g}$ valued fields ω, ϕ of degrees 1,2, respectively. $\mathfrak{t}$ acts on $\mathfrak{w}$ according to (49)–(51) below.

$k, d, j(\xi), l(\xi)$ can be organized into two $N=1$ topological superderivation

$$H = k - \theta d, \tag{3}$$

$$I(\xi) = j(\xi) + \theta l(\xi), \quad \xi \in \mathfrak{g}. \tag{4}$$

The Lie algebra structure of $\mathfrak{t}$ is compatible with the underlying $N=1$ topological supersymmetry, since the commutation relations of $\mathfrak{t}$ can be written in terms of the superderivations $H, I(\xi)$. Similarly, ω, ϕ can be organized into the $\mathfrak{g}$ valued superfield

$$W = \omega + \theta(\phi - \tfrac{1}{2}[\omega, \omega]). \tag{5}$$

The action of $\mathfrak{t}$ on $\mathfrak{w}$ can be written in terms of the superderivations $H, I(\xi)$ and the superfield W in a manifestly $N=1$ supersymmetric way.

Analogously, in $N=2$ cohomological topological field theory, $\mathfrak{t}$ is generated by seven graded derivations u_A, $A=1,2$, t_{AB}, $A, B = 1, 2$, symmetric in A, B, k, d_A, $A=1,2$, $j(\xi)$, $j_A(\xi)$, $A=1,2$, $l(\xi)$, $\xi \in \mathfrak{g}$, of degrees $-1, 0, 0, 1, -2, -1, 0$, respectively, obeying the graded commutation relations (40)–(44) below. The u_A are a sort of homotopy operators and constrain the cohomology of $\mathfrak{f}$, defined shortly, to an important extent. The t_{AB} and k are the generators of the internal $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$ symmetry Lie algebra of $\mathfrak{t}$. The d_A are the nilpotent topological charges. $j(\xi), j_A(\xi), l(\xi)$ describe the action of the symmetry Lie algebra $\mathfrak{g}$ on fields. The elements $\alpha \in \mathfrak{f}$ are classified into the eigenspaces $\mathfrak{f}^{n,p}$, $n \in \mathbb{N}$, $p \in \mathbb{Z}$, of the invariants c, k of the internal algebra $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$. The $N=2$ basic type n, p cohomology of $\mathfrak{f}$ is defined by

$$j(\xi)\alpha = 0, \quad j_A(\xi)\alpha = 0, \quad l(\xi)\alpha = 0, \quad \xi \in \mathfrak{g}, \quad d_A\alpha = 0, \tag{6}$$

$$\begin{aligned} &\alpha \equiv \alpha + \tfrac{1}{2}\epsilon^{KL} d_K d_L \beta, \quad \beta \in \mathfrak{f}^{n,p-2}, \quad j(\xi)\beta = 0, \\ &\quad j_A(\xi)\beta = 0, \quad l(\xi)\beta = 0, \quad \xi \in \mathfrak{g}, \end{aligned} \tag{7}$$

where $\alpha \in \mathfrak{f}^{n,p}$. It is possible to show, using the basic relation $[d_A, u_B] = \frac{1}{2}(t_{AB} + \epsilon_{AB}k)$, that this cohomology is trivial for $p \neq \pm n + 1$.

The $N = 2$ Weil algebra $\mathfrak{w}$, entering the definition of $N = 2$ equivariant cohomology, is generated by four $\mathfrak{g}$ valued fields ω_A, $A = 1, 2$, ϕ_{AB}, $A, B = 1, 2$, symmetric in A, B, γ, ρ_A, $A = 1, 2$, of degrees 1,2,2,3, respectively. $\mathfrak{t}$ acts on $\mathfrak{w}$ according to (55)–(57).

$u_A, t_{AB}, k, d_A, j(\xi), j_A(\xi), l(\xi)$ can be organized into two $N = 2$ topological superderivation

$$H_A = u_A + \tfrac{1}{2}\theta^K(t_{AK} - \epsilon_{AK}k) - \tfrac{1}{2}\epsilon_{KL}\theta^K\theta^L d_A, \tag{8}$$

$$I(\xi) = j(\xi) + \theta^K j_K(\xi) + \tfrac{1}{2}\epsilon_{KL}\theta^K\theta^L l(\xi), \quad \xi \in \mathfrak{g}. \tag{9}$$

The Lie algebra structure of $\mathfrak{t}$ is compatible with the underlying $N = 2$ topological supersymmetry, since the commutation relations of $\mathfrak{t}$ can be written in terms of the superderivations $H_A, I(\xi)$. Similarly, $\omega_A, \phi_{AB}, \gamma, \rho_A$ can be organized into the $\mathfrak{g}$ valued superfield

$$\begin{aligned} W_A = {} & \omega_A + \theta^K(\phi_{AK} + \epsilon_{AK}\gamma - \tfrac{1}{2}[\omega_A, \omega_K]) + \tfrac{1}{2}\epsilon_{MN}\theta^M\theta^N(-2\rho_A - \epsilon^{KL}[\omega_K, \phi_{AL}] \\ & +[\omega_A, \gamma] + \tfrac{1}{6}\epsilon^{KL}[\omega_K, [\omega_L, \omega_A]]). \end{aligned} \tag{10}$$

The action of $\mathfrak{t}$ on $\mathfrak{w}$ can be written in terms of the superderivations $H_A, I(\xi)$ and the superfield W_A in a manifestly $N = 2$ supersymmetric way.

In the first part of this paper, we study the topological algebra $\mathfrak{t}$ and the Weil algebra $\mathfrak{w}$ abstractly both in the $N = 1$ and in the $N = 2$ case. We show that their structure is essentially dictated by rather general requirements of closure and topological supersymmetry, which can be defined for any value of N. In the second part of the paper, we define basic and equivariant cohomology, abstract connections and the Weil homomorphism both in the $N = 1$ and in the $N = 2$ case and study some of their properties. Finally, in the third part of the paper, we study the cohomology of manifolds carrying a right group action and show that, in this important case, the $N = 2$ type $(k, k + 1)$ basic cohomology is isomorphic to the tensor product of the $N = 1$ degree k basic cohomology and the completely symmetric tensor space $\bigvee^{k-1} \mathbb{R}^2$ and that the affine spaces of $N = 2$ and $N = 1$ connections are isomorphic.

Throughout the paper, we stress the role of topological supersymmetry, also because we feel that, on this score, confusing claims have appeared in the literature. This has allowed us to discover the derivations u_A and k introduced above, which are not mentioned in Ref. [31], but which are required by $N = 2$ topological supersymmetry and constrain structurally the $N = 2$ cohomology.

The definition of $N = 2$ basic cohomology given above is more general than that used in Ref. [31], which is limited to the important case where $N = 1$. In our judgement, this definition is more appropriate, yielding the aforementioned fundamental relation between the $N = 1$ and $N = 2$ basic cohomologies of manifolds with a right group action.

This paper is organized as follows. We have tried to highlight the similarities and the differences of the $N = 1$ and $N = 2$ cases in order to show in what sense the latter is a generalization of the former. In Section 2, we briefly review the basic facts of the theory of superalgebras and supermodules. In Section 3, we introduce the $N = 1$ and $N = 2$ topological algebras. In Section 4, we introduce the $N = 1$ and $N = 2$ Weil algebras. In Section 5, we define the relevant notions of $N = 1$ and $N = 2$ (basic) cohomology. In Section 6, we study the $N = 1$ and $N = 2$ Weil superoperations and their (basic) cohomology and derive the relation between $N = 1$ and $N = 2$ cohomology. In Section 7, we define $N = 1$ and $N = 2$ abstract connections, equivariant cohomology and the related Weil homomorphism. In Section 8, we apply our algebraic setup to study the $N = 1$ and $N = 2$ (basic) cohomology of manifolds carrying a right group action and work out the relation between $N = 1$ and $N = 2$ cohomologies. Finally, Section 9 outlines future lines of inquiry.

2. Superalgebras and supermodules

2.1. $\mathbb{Z}$ graded algebras and the corresponding superalgebras

We begin by stipulating the following.

All the vector spaces, algebras and modules considered in this paper are real.

If s is a $\mathbb{Z}$ graded space, we denote by s^k the subspace of s of degree $k \in \mathbb{Z}$. If $\mathsf{s} = \mathsf{s}^0$, s is called ungraded.

Let $N \in \mathbb{N}$. Let θ^A, $A = 1, \dots, N$, be a N-tuple of Grassmann odd generators which are conventionally assigned degree -1:

$$\theta^A\theta^B + \theta^B\theta^A = 0, \quad A, B = 1, \dots, N; \quad \deg\theta^A = -1, \quad A = 1, \dots, N. \tag{11}$$

The θ^A generate a Grassmann algebra $\Lambda_N[\theta]$. The derivatives $\partial_A = \partial/\partial\theta_A$ are degree $+1$ graded derivations on $\Lambda_N[\theta]$.

Let v be a $\mathbb{Z}$ graded space. The N superspace V_N associated to v is the graded tensor product space

$$\mathsf{V}_N = \Lambda_N[\theta]\hat{\otimes}\mathsf{v} \tag{12}$$

with the canonical $\mathbb{Z}$ grading. Given a $\mathbb{Z}$ graded algebra a, one can define the N superalgebra A_N in similar fashion. Note that ∂_A extends to a degree $+1$ graded linear operator on V_N and to a degree $+1$ graded derivation on A_N.

Definition 1. A $\mathbb{Z}$ graded space x is called an N superspace if:
1. there is a $\mathbb{Z}$ graded space v such that x is isomorphic to a subspace of V_N invariant under all ∂_A;
2. there is a minimal subspace x_* of x such that $\mathsf{x} = \Lambda_N[\partial]\mathsf{x}_*$, where $\Lambda_N[\partial]$ is the Grassmann algebra of polynomials of the derivations ∂_A.

The notion of N superalgebra can be given for a $\mathbb{Z}$ graded algebra a in analogous fashion.

$\mathsf{x}_*(\mathsf{a}_*)$ is the generating subspace (subalgebra) of $\mathsf{x}(\mathsf{a})$.

Definition 2. A $\mathbb{Z}$ graded left module m of a $\mathbb{Z}$ graded algebra a is an N left a supermodule if:
1. a is an N superalgebra;
2. m is an N superspace;
3. the ∂_A are graded derivations with respect to the module multiplication.

The notion of N supermodule algebra can be given in analogous fashion.

In this paper, we are mostly concerned with $\mathbb{Z}$ graded Lie algebras. A $\mathbb{Z}$ graded Lie algebra l is a $\mathbb{Z}$ graded algebra whose product is graded antisymmetric and satisfies the graded Jacobi identity.

For a $\mathbb{Z}$ graded Lie algebra l, a $\mathbb{Z}$ graded left l module algebra m with unity 1 is derivative if the action of l on m obeys the graded Leibniz rule.

2.2. The $N = 1, 2$ cases

In this paper, we concentrate on the cases $N = 1, 2$. In this subsection, we introduce notation suitable for these special N values.

Let a be a $\mathbb{Z}$ graded algebra.

Let $N = 1$. In this case, one can set $\theta^1 = \theta$ for simplicity. If $X \in \mathsf{A}_1^p$ for some $p \in \mathbb{Z}$, then X is of the form

$$X = x + \theta\tilde{x} \tag{13}$$

with $x \in \mathsf{a}^p$ and $\tilde{x} \in \mathsf{a}^{p+1}$. Note that

$$x = X|_{\theta=0}. \tag{14}$$

Denoting $\partial = \partial/\partial\theta$, we define

$$\tilde{X} = \partial X. \tag{15}$$

Clearly, $\tilde{X} \in \mathsf{A}_1^{p+1}$. Indeed,

$$\tilde{X} = \tilde{x}. \tag{16}$$

Let $N = 2$. If $X \in \mathsf{A}_2^p$ for some $p \in \mathbb{Z}$, then X is of the form

$$X = x + \theta^A x_{,A} + \tfrac{1}{2}\epsilon_{KL}\theta^K\theta^L\tilde{x} \tag{17}$$

with $x \in \mathsf{a}^p$, $x_{,A} \in \mathsf{a}^{p+1}$ and $\tilde{x} \in \mathsf{a}^{p+2}$.[1] Note that

$$x = X|_{\theta=0}. \tag{18}$$

Denoting $\partial_A = \partial/\partial\theta^A$, we define

$$X_{,A} = \partial_A X. \tag{19}$$

[1] The totally antisymmetric symbols ϵ_{AB}, ϵ^{AB} are normalized so that $|\epsilon_{12}| = |\epsilon^{12}| = 1$ and $\epsilon^{AK}\epsilon_{KB} = \epsilon_{BK}\epsilon^{KA} = \delta^A_B$.

Clearly, $X_{,A} \in \mathsf{A}_2^{p+1}$. Indeed,

$$X_{,A} = x_{,A} + \epsilon_{AK}\theta^K \tilde{x}. \tag{20}$$

So,

$$x_{,A} = X_{,A}|_{\theta=0}. \tag{21}$$

Finally, we set

$$\tilde{X} = \tfrac{1}{2}\epsilon^{KL}\partial_K \partial_L X. \tag{22}$$

Clearly, $\tilde{X} \in \mathsf{A}_2^{p+2}$, as

$$\tilde{X} = \tilde{x}. \tag{23}$$

3. Fundamental superstructures

In this section, we shall introduce the fundamental $N = 1$ and $N = 2$ superstructures. We shall present them without attempting a derivation from a simpler, more basic set of axioms. Though this would be desirable, it would bring us to far afield. Their justification lies ultimately in the applications they have in differential geometry and, in the infinite dimensional case, in topological quantum field theory.

Let $\mathfrak{g}$ be an ungraded Lie algebra.

3.1. *The fundamental $N = 1$ superstructure*

Definition 3. The fundamental $N = 1$ superstructure t of $\mathfrak{g}$ is the $N = 1$ Lie superalgebra defined by

1. t is generated by $H, I(\xi), \xi \in \mathfrak{g}$, where $H \in \mathsf{t}^0$ and $I : \mathfrak{g} \mapsto \mathsf{t}^{-1}$ is a linear map;
2. the following commutation relations hold:

$$[H, H] = 0, \quad [H, \tilde{H}] = \tilde{H}, \quad [\tilde{H}, \tilde{H}] = 0, \tag{24a–c}$$

$$[I(\xi), I(\eta)] = 0, \quad [I(\xi), \tilde{I}(\eta)] = I([\xi, \eta]),$$

$$[\tilde{I}(\xi), \tilde{I}(\eta)] = \tilde{I}([\xi, \eta]), \quad \xi, \eta \in \mathfrak{g}, \tag{25a–c}$$

$$[H, I(\xi)] = -I(\xi), \quad [H, \tilde{I}(\xi)] = 0,$$

$$[\tilde{H}, I(\xi)] = -\tilde{I}(\xi), \quad [\tilde{H}, \tilde{I}(\xi)] = 0, \quad \xi \in \mathfrak{g}. \tag{26a–d}$$

It is straightforward to verify that the above commutation relations fulfil the graded antisymmetry and Jacobi identities.

The components $h, \tilde{h}, i(\xi), \tilde{i}(\xi), \xi \in \mathfrak{g}$ satisfy relations (24)–(26) and thus are the generators of a $\mathbb{Z}$ graded Lie algebra isomorphic to t. Thus, t could be defined alternatively in this latter way. The definition given above shows that t is indeed a $N = 1$ Lie superalgebra.

More customarily, one sets

$$k = h, \quad d = -\tilde{h}, \tag{27a,b}$$

$$j(\xi) = i(\xi), \qquad l(\xi) = \tilde{i}(\xi), \quad \xi \in \mathfrak{g}. \tag{28a,b}$$

From (24)–(26), one sees that k, d, j and l satisfy the relations

$$[k, k] = 0, \tag{29}$$

$$[k, d] = d, \quad [k, j(\xi)] = -j(\xi),$$

$$[k, l(\xi)] = 0, \quad \xi \in \mathfrak{g}, \tag{30a–c}$$

$$[d, d] = 0, \tag{31}$$

$$[d, j(\xi)] = l(\xi), \qquad [d, l(\xi)] = 0, \qquad \xi \in \mathfrak{g}, \tag{32a,b}$$

$$[j(\xi), j(\eta)] = 0, \quad [j(\xi), l(\eta)] = j([\xi, \eta]),$$

$$[l(\xi), l(\eta)] = l([\xi, \eta]), \qquad \xi, \eta \in \mathfrak{g}. \tag{33a–c}$$

Note that, by (29), k generates an ungraded Lie subalgebra

$$\mathfrak{i} \simeq \mathbb{R} \tag{34}$$

of $\mathfrak{t}$. $\mathfrak{i}$ is called the internal symmetry algebra of the fundamental $N = 1$ superstructure $\mathfrak{t}$.

3.2. *The fundamental $N = 2$ superstructure*

Definition 4. The fundamental $N = 2$ superstructure $\mathfrak{t}$ of $\mathfrak{g}$ is the $N = 2$ Lie superalgebra defined by:
1. $\mathfrak{t}$ is generated by H_A, $A = 1, 2$, $I(\xi)$, $\xi \in \mathfrak{g}$, where $H_A \in \mathfrak{t}^{-1}$ and $I : \mathfrak{g} \mapsto \mathfrak{t}^{-2}$ is a linear map;
2. the following commutations relations hold:

$$[H_A, H_B] = 0, \quad [H_A, H_{B,C}] = \epsilon_{AB} H_C,$$

$$[H_A, \tilde{H}_B] = -H_{A,B}, \quad [H_{A,C}, H_{B,D}] = \epsilon_{AB} H_{C,D} - \epsilon_{DC} H_{B,A},$$

$$[H_{A,C}, \tilde{H}_B] = -\epsilon_{BC} \tilde{H}_A, \quad [\tilde{H}_A, \tilde{H}_B] = 0. \tag{35e–f}$$

$$[I(\xi), I(\eta)] = 0, \quad [I(\xi), I_{,A}(\eta)] = 0,$$

$$[I(\xi), \tilde{I}(\eta)] = I([\xi, \eta]), \quad [I_{,A}(\xi), I_{,B}(\eta)] = \epsilon_{AB} I([\xi, \eta]),$$

$$[I_{,A}(\xi), \tilde{I}(\eta)] = I_{,A}([\xi, \eta]), \quad [\tilde{I}(\xi), \tilde{I}(\eta)] = \tilde{I}([\xi, \eta]), \qquad \xi, \eta \in \mathfrak{g}. \tag{36e–f}$$

$$[H_A, I(\xi)] = 0, \quad [H_A, I_{,B}(\xi)] = \epsilon_{AB} I(\xi),$$

$$[H_A, \tilde{I}(\xi)] = 0, \quad [H_{A,B}, I(\xi)] = \epsilon_{AB} I(\xi),$$

$$[H_{A,C}, I_{,B}(\xi)] = \epsilon_{AB} I_{,C}(\xi), \quad [H_{A,B}, \tilde{I}(\xi)] = 0,$$

$$[\tilde{H}_A, I(\xi)] = -I_{,A}(\xi), \quad [\tilde{H}_A, I_{,B}(\xi)] = \epsilon_{AB}\tilde{I}(\xi),$$

$$[\tilde{H}_A, \tilde{I}(\xi)] = 0. \qquad \xi \in \mathfrak{g}. \tag{37a–i}$$

It is straightforward to verify that the above commutation relations fulfil the graded anti-symmetry and Jacobi identities.

The components $h_A, h_{A,B}, \tilde{h}_A, i(\xi), i_{,A}(\xi), \tilde{i}(\xi), \xi \in \mathfrak{g}$ satisfy relations (35)–(37) and thus are the generators of a $\mathbb{Z}$ graded Lie algebra isomorphic to t. Thus, t could be defined alternatively in this latter way. The definition given above has the advantage of showing that t is indeed a $N = 2$ Lie superalgebra.

To make contact with Ref. [31], one sets

$$t_{AB} = h_{A,B} + h_{B,A}, \quad k = \epsilon^{KL} h_{K,L}, \quad u_A = h_A, \quad d_A = -\tilde{h}_A, \tag{38a–d}$$

$$j(\xi) = i(\xi), \quad j_A(\xi) = i_{,A}(\xi), \quad l(\xi) = \tilde{i}(\xi), \quad \xi \in \mathfrak{g}. \tag{39a–c}$$

From (35)–(37), one sees that $t_{AB}, k, u_A, d_A, j, j_A$ and l satisfy the relations

$$[t_{AC}, t_{BD}] = \epsilon_{AB} t_{CD} + \epsilon_{CB} t_{AD} + \epsilon_{AD} t_{BC} + \epsilon_{CD} t_{BA}, \quad [k, t_{AB}] = 0, \quad [k, k] = 0, \tag{40a–c}$$

$$[t_{AC}, u_B] = \epsilon_{AB} u_C + \epsilon_{CB} u_A, \quad [k, u_A] = -u_A,$$

$$[t_{AC}, d_B] = \epsilon_{AB} d_C + \epsilon_{CB} d_A, \quad [k, d_A] = d_A,$$

$$[t_{AB}, j(\xi)] = 0, \quad [k, j(\xi)] = -2j(\xi),$$

$$[t_{AC}, j_B(\xi)] = \epsilon_{AB} j_C(\xi) + \epsilon_{CB} j_A(\xi), \quad [k, j_A(\xi)] = -j_A(\xi),$$

$$[t_{AB}, l(\xi)] = 0, \quad [k, l(\xi)] = 0, \qquad \xi \in \mathfrak{g}, \tag{41a–j}$$

$$[u_A, u_B] = 0, \quad [d_A, u_B] = \tfrac{1}{2}(t_{AB} + \epsilon_{AB} k), \quad [d_A, d_B] = 0, \tag{42a–c}$$

$$[u_A, j(\xi)] = 0, \quad [u_A, j_B(\xi)] = \epsilon_{AB} j(\xi),$$

$$[u_A, l(\xi)] = 0, \quad [d_A, j(\xi)] = j_A(\xi),$$

$$[d_A, j_B(\xi)] = -\epsilon_{AB} l(\xi), \quad [d_A, l(\xi)] = 0, \qquad \xi \in \mathfrak{g}, \tag{43a–f}$$

$$[j(\xi), j(\eta)] = 0, \quad [j(\xi), j_A(\eta)] = 0,$$

$$[j(\xi), l(\eta)] = j([\xi, \eta]), \quad [j_A(\xi), j_B(\eta)] = \epsilon_{AB} j([\xi, \eta]),$$

$$[j_A(\xi), l(\eta)] = j_A([\xi, \eta]), \quad [l(\xi), l(\eta)] = l([\xi, \eta]), \qquad \xi, \eta \in \mathfrak{g}. \tag{44a–f}$$

Note that, from (40), t_{AB}, k generate an ungraded Lie subalgebra

$$\mathsf{i} \simeq \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R} \tag{45}$$

of t. i is called the internal symmetry algebra of the $N = 2$ fundamental superstructure t and plays an important role.

4. The Weil algebra

In this section, we shall introduce the $N=1$ and $N=2$ Weil algebras. As we did in the case of the fundamental $N=1$ and $N=2$ superstructures, we shall not attempt a derivation from a simpler, more basic set of axioms. Again, their justification lies ultimately in the applications they have in differential geometry and in topological quantum field theory.

Let $\mathfrak{g}$ be an ungraded Lie algebra.

4.1. The $N=1$ case

Definition 5. The $N=1$ Weil algebra w of $\mathfrak{g}$ is the $N=1$ left supermodule algebra with unity of the $N=1$ fundamental superstructure t of $\mathfrak{g}$ (cf. Section 3.1) defined by the following properties:
1. w is derivative;
2. w is generated by 1, $W(\mu)$, $\mu\in\mathfrak{g}^\vee$, where $W:\mathfrak{g}^\vee\mapsto\mathsf{w}^1$ is a linear map;
3. the following relations hold:

$$HW=W,\quad H\tilde{W}=2\tilde{W},$$
$$\tilde{H}W=-\tilde{W},\quad \tilde{H}\tilde{W}=0, \tag{46a–d}$$
$$I(\xi)W=\xi,\quad I(\xi)\tilde{W}=-[\xi,W],$$
$$\tilde{I}(\xi)W=-[\xi,W],\quad \tilde{I}(\xi)\tilde{W}=-[\xi,\tilde{W}],\qquad \xi\in\mathfrak{g}, \tag{47a–d}$$

where W is viewed as an element of $\mathsf{w}\otimes\mathfrak{g}$.

It is straightforward to verify that the above relations do indeed define a $\mathbb{Z}$ graded module of t.

Note that the components 1, w, $\tilde{w}$ and the component derivations h, $\tilde{h}$, $i(\xi)$, $\tilde{i}(\xi)$, $\xi\in\mathfrak{g}$, satisfy relations (46) and (47). Hence, 1, w, $\tilde{w}$ generate a derivative $\mathbb{Z}$ graded left t module algebra with unity isomorphic to w. Thus, w could be defined alternatively in this latter way. The definition given above shows that w is indeed an $N=1$ t left Lie module superalgebra.

In the standard treatment, w is usually presented as follows. Define

$$\omega=w,\quad \phi=\tilde{w}+\tfrac{1}{2}[w,w]. \tag{48a,b}$$

Then, one has

$$k\omega=\omega,\quad k\phi=2\phi, \tag{49a,b}$$

$$d\omega=\phi-\tfrac{1}{2}[\omega,\omega],\quad d\phi=-[\omega,\phi], \tag{50a,b}$$

$$j(\xi)\omega=\xi,\quad j(\xi)\phi=0,\quad l(\xi)\omega=-[\xi,\omega],\quad l(\xi)\phi=-[\xi,\phi],\quad \xi\in\mathfrak{g}, \tag{51a,b}$$

where k, d, j, l are given by (27) and (28). Note that ω is just another name for w. ϕ is by construction ‘horizontal’, i.e. satisfying (51b).

4.2. The $N=2$ case

Definition 6. The $N=2$ Weil algebra w of $\mathfrak{g}$ is the $N=2$ left supermodule algebra with unity of the $N=2$ fundamental superstructure t of $\mathfrak{g}$ (cf. Section 3.2) defined by the following properties:

1. w is derivative;
2. w is generated by $1, W_A(\mu)$, $A=1,2$, $\mu\in\mathfrak{g}^\vee$, where $W_A:\mathfrak{g}^\vee\mapsto \mathsf{w}^1$ is a linear map;
3. the following relations hold:

$$\begin{aligned}
&H_AW_B=0,\quad H_AW_{B,C}=-\epsilon_{BC}W_A,\\
&H_A\tilde W_B=-W_{A,B}-W_{B,A},\quad H_{A,C}W_B=-\epsilon_{BC}W_A,\\
&H_{A,C}W_{B,D}=\epsilon_{CB}W_{A,D}-\epsilon_{DC}W_{B,A},\quad H_{A,C}\tilde W_B=-\epsilon_{BC}\tilde W_A-\epsilon_{AC}\tilde W_B,\\
&\tilde H_AW_B=-W_{B,A},\quad \tilde H_AW_{B,C}=\epsilon_{AC}\tilde W_B,\quad \tilde H_A\tilde W_B=0,
\end{aligned}\tag{52a–i}$$

$$\begin{aligned}
&I(\xi)W_A=0,\quad I(\xi)W_{A,B}=\epsilon_{AB}\xi,\\
&I(\xi)\tilde W_A=-[\xi,W_A],\quad I_{,A}(\xi)W_B=\epsilon_{AB}\xi,\\
&I_{,A}(\xi)W_{B,C}=-\epsilon_{AC}[\xi,W_B],\qquad I_{,A}(\xi)\tilde W_B=-[\xi,W_{B,A}],\\
&\tilde I(\xi)W_A=-[\xi,W_A],\quad \tilde I(\xi)W_{A,B}=-[\xi,W_{A,B}],\\
&\tilde I(\xi)\tilde W_A=-[\xi,\tilde W_A],\qquad \xi\in\mathfrak{g}.
\end{aligned}\tag{53a–i}$$

where W_A is viewed as an element of $\mathsf{w}\otimes\mathfrak{g}$.

It is straightforward to verify that the above relations do indeed define a $\mathbb{Z}$ graded module of t.

Note that the components $1, w_A, w_{A,B}, \tilde w_A$ and the component derivations $h_A, h_{A,B}$, $\tilde h_A, i(\xi), i_{,A}(\xi), \tilde i(\xi)$, $\xi\in\mathfrak{g}$, satisfy relations (52) and (53). Hence, $1, w_A, w_{A,B}, \tilde w_A$ generate a derivative $\mathbb{Z}$ graded left t module algebra with unity isomorphic to w. Thus, w could be defined alternatively in this latter way. The definition given above shows that w is indeed a $N=2\mathsf{t}$ left Lie module superalgebra.

To make contact with Ref. [31], we shall present w as follows. Define

$$\begin{aligned}
&\omega_A=w_A,\qquad \phi_{AB}=\tfrac12(w_{A,B}+w_{B,A}+[w_A,w_B]),\\
&\gamma=-\tfrac12\epsilon^{KL}w_{K,L},\quad \rho_A=-\tfrac12\tilde w_A-\tfrac12\epsilon^{KL}[w_K,w_{A,L}]-\tfrac16\epsilon^{KL}[w_K,[w_L,w_A]].
\end{aligned}\tag{54a–d}$$

Then, one has

$$\begin{aligned}
&t_{AC}\omega_B=\epsilon_{AB}\omega_C+\epsilon_{CB}\omega_A, && k\omega_A=\omega_A,\\
&t_{AC}\phi_{BD}=\epsilon_{AB}\phi_{CD}+\epsilon_{CB}\phi_{AD}+\epsilon_{AD}\phi_{BC}+\epsilon_{CD}\phi_{BA}, && k\phi_{AB}=2\phi_{AB},\\
&t_{AB}\gamma=0, && k\gamma=2\gamma,\\
&t_{AC}\rho_B=\epsilon_{AB}\rho_C+\epsilon_{CB}\rho_A, && k\rho_A=3\rho_A,
\end{aligned}\tag{55a–h}$$

$$\begin{aligned}
&u_A\omega_B = 0, && u_A\phi_{BC} = 0,\\
&u_A\gamma = -\omega_A, && u_A\rho_B = \phi_{AB},\\
&d_A\omega_B = -\tfrac{1}{2}[\omega_A,\omega_B] + \phi_{AB} - \epsilon_{AB}\gamma, && d_A\phi_{BC} = -[\omega_A,\phi_{BC}] + \epsilon_{AB}\rho_C\\
&d_A\gamma = -\tfrac{1}{2}[\omega_A,\gamma] + \rho_A && \qquad +\epsilon_{AC}\rho_B,\\
&\qquad + \tfrac{1}{2}\epsilon^{KL}[\omega_K,\phi_{LA} - \tfrac{1}{6}[\omega_L,\omega_A]], && d_A\rho_B = -[\omega_A,\rho_B]\\
& && \qquad -\tfrac{1}{2}\epsilon^{KL}[\phi_{KA},\phi_{LB}],
\end{aligned} \tag{56a–h}$$

$$\begin{aligned}
&j(\xi)\omega_A = 0, && j(\xi)\phi_{AB} = 0,\\
&j(\xi)\gamma = \xi, && j(\xi)\rho_A = 0,\\
&j_A(\xi)\omega_B = \epsilon_{AB}\xi, && j_A(\xi)\phi_{BC} = 0,\\
&j_A(\xi)\gamma = -\tfrac{1}{2}[\xi,\omega_A], && j_A(\xi)\rho_B = 0,\\
&l(\xi)\omega_A = -[\xi,\omega_A], && l(\xi)\phi_{AB} = -[\xi,\phi_{AB}],\\
&l(\xi)\gamma = -[\xi,\gamma], && l(\xi)\rho_A = -[\xi,\rho_A], \qquad \xi\in\mathfrak{g},
\end{aligned} \tag{57a–l}$$

where $t_{A,B}, k, u_A, d_A, j, j_A, l$ are given by (38) and (39). Note that ω_A is just another name for w_A. γ contains the information about $\tilde{h}_A w_B$ not exhausted by ϕ_{AB}. By construction ϕ_{AB} and ρ_A are 'horizontal', i.e. satisfy (57b,d,f,h).

5. Superoperations and their cohomologies

Let $\mathfrak{g}$ be an ungraded Lie algebra.

5.1. $N = 1$ superoperations and their cohomologies

Definition 7. a is called an $N = 1$ $\mathfrak{g}$ superoperation if:
1. a is a $\mathbb{Z}$ graded left module algebra of the fundamental $N = 1$ superstructure t of $\mathfrak{g}$ (cf. Section 3.1);
2. the action of t on a is derivative;
3. a is completely reducible under the internal symmetry algebra i of t (cf. Section 3.1), the spectrum of the invariant k of i is integer and the eigenspace a^p of k of the eigenvalue $p \in \mathbb{Z}$ is precisely the degree p subspace of a.

So, a is acted upon by four graded derivations $h, \tilde{h}, i(\xi), \tilde{i}(\xi), \xi \in \mathfrak{g}$, of degree $0, +1, -1, 0$, respectively, satisfying relations (24)–(26), or, equivalently, by four graded derivations $k, d, j(\xi), l(\xi), \xi \in \mathfrak{g}$, of degree $0, +1, -1, 0$, respectively, satisfying relations (29)–(33), the two sets of derivations being related as in (27) and (28).

Proposition 1. *If* $\mathsf{a}^{(r)}, r = 1, 2,$ *are two* $N = 1$ $\mathfrak{g}$ *superoperations, then their graded tensor product* $\mathsf{a} = \mathsf{a}^{(1)}\hat{\otimes}\mathsf{a}^{(2)}$ *is also an* $N = 1$ $\mathfrak{g}$ *superoperation.*

Proof. Indeed a satisfies the conditions stated in Definition 7. □

Let a be an $N = 1\mathfrak{g}$ superoperation.

The pair (a, d) is an ordinary differential complex, as the graded derivation d has degree $+1$ and $[d, d] = 0$. Its cohomology $H^*(\mathsf{a})$, defined in the usual way by

$$H^p(\mathsf{a}) = (\ker d \cap \mathsf{a}^p)/d\mathsf{a}^{p-1}, \quad p \in \mathbb{Z}, \tag{58}$$

is the ordinary cohomology of the superoperation. Define

$$\mathsf{a}_{\mathrm{basic}} = \bigcap_{\xi\in\mathfrak{g}} \ker j(\xi) \cap \ker l(\xi). \tag{59}$$

By (32), $\mathsf{a}_{\mathrm{basic}}$ is d invariant. So, $(\mathsf{a}_{\mathrm{basic}}, d)$ is also a differential complex. Its cohomology $H^*_{\mathrm{basic}}(\mathsf{a})$

$$H^p_{\mathrm{basic}}(\mathsf{a}) = (\ker d \cap \mathsf{a}^p_{\mathrm{basic}})/d\mathsf{a}^{p-1}_{\mathrm{basic}}, \quad p \in \mathbb{Z}, \tag{60}$$

is the basic cohomology of the superoperation.

Proposition 2. *Each nonzero (basic) cohomology class of degree p defines a one-dimensional representation of the internal Lie algebra i of invariant p.*

Proof. Set $k[x] = [kx] = p[x]$ for $[x] \in H^p(\mathsf{a})([x] \in H^p_{\mathrm{basic}}(\mathsf{a}))$ with arbitrary representative $x \in \mathsf{a}^p (x \in \mathsf{a}^p_{\mathrm{basic}})$. □

Though the above proposition is trivial, it is nevertheless interesting because of its non-trivial generalization to higher N.

5.2. $N = 2$ superoperations and their cohomologies

Definition 8. a is called an $N = 2\mathfrak{g}$ superoperation if:

1. a is a $\mathbb{Z}$ graded left module algebra of the fundamental $N = 2$ superstructure t of $\mathfrak{g}$ (cf. Section 3.2);
2. the action of t on a is derivative;
3. a is completely reducible under the internal symmetry algebra i of t (cf. Section 3.2), the spectrum of the invariant k of i is integer and the eigenspace a^p of k of the eigenvalue $p \in \mathbb{Z}$ is precisely the degree p subspace of a.

So, a is acted upon by six graded derivations $h_A, h_{A,B}, \tilde{h}_A, i(\xi), i_{,A}(\xi), \tilde{i}(\xi), \xi \in \mathfrak{g}$, of degree $-1, 0, +1, -2, -1, 0$, respectively, satisfying relations (35)–(37), or, equivalently, by seven graded derivations $t_{AB}, k, u_A, d_A, j(\xi), j_A(\xi), l(\xi), \xi \in \mathfrak{g}$, of degree $0, 0, -1, +1, -2, -1, 0$, respectively, satisfying relations (40)–(44), the two sets of derivations being related as in (38) and (39).

Besides k, i possesses another invariant, namely

$$c = -\tfrac{1}{8}\epsilon^{KL}\epsilon^{MN}t_{KM}t_{LN}. \tag{61}$$

An irreducible representation of i is completely characterized up to equivalence by the values of c and k, which we parametrize as $\frac{1}{4}(n^2-1)$ and p, respectively, where $n \in \mathbb{N}$ and $p \in \mathbb{Z}$. n is nothing but the dimension of the representation. Being completely reducible under i, a organizes into irreducible representations of i. We denote by $\mathsf{a}^{n,p}$ the eigenspace of c, k of eigenvalues $\frac{1}{4}(n^2-1)$, p, respectively. It follows that a has a finer grading than the original one.

Proposition 3. *If* $\mathsf{a}^{(r)}$, $r=1,2$, *are two* $N=2$ $\mathfrak{g}$ *superoperations, then their graded tensor product* $\mathsf{a}=\mathsf{a}^{(1)}\hat{\otimes}\mathsf{a}^{(2)}$ *is also an* $N=2$ $\mathfrak{g}$ *superoperation.*

Proof. Indeed a satisfies the conditions stated in Definition 8. □

Let a be an $N=2\mathfrak{g}$ superoperation.

The graded derivations d_A have degree $+1$ and satisfy $[d_A, d_B]=0$. So, one may define a double differential complex (a, d_A). We do not define cohomology in the usual way, as the standard definition would not be covariant with respect to i. Instead, we propose the following definition generalizing that of Ref. [31]. The ordinary cohomology $H^*(\mathsf{a})$ is labelled by the values of the invariants c, k of i and is defined as

$$H^{n,p}(\mathsf{a}) = (\cap_{A=1,2}\ker d_A \cap \mathsf{a}^{n,p})/\tfrac{1}{2}\epsilon^{KL}d_K d_L \mathsf{a}^{n,p-2}, \qquad (n,p)\in\mathbb{N}\times\mathbb{Z}. \tag{62}$$

The basic subspace of a is defined as

$$\mathsf{a}_{\mathrm{basic}} = \bigcap_{\xi\in\mathfrak{g}} \ker j(\xi) \cap \cap_{A=1,2}\ker j_A(\xi) \cap \ker l(\xi). \tag{63}$$

Using (43d–f), one can show that $\mathsf{a}_{\mathrm{basic}}$ is d_A invariant. So, $(\mathsf{a}_{\mathrm{basic}}, d_A)$ is also a double differential complex. Its cohomology $H^*_{\mathrm{basic}}(\mathsf{a})$ is defined

$$H^{n,p}_{\mathrm{basic}}(\mathsf{a}) = (\cap_{A=1,2}\ker d_A \cap \mathsf{a}^{n,p}_{\mathrm{basic}})/\tfrac{1}{2}\epsilon^{KL}d_K d_L \mathsf{a}^{n,p-2}_{\mathrm{basic}}, \qquad (n,p)\in\mathbb{N}\times\mathbb{Z}, \tag{64}$$

where $\mathsf{a}^{n,p}_{\mathrm{basic}} = \mathsf{a}^{n,p}\cap\mathsf{a}_{\mathrm{basic}}$, and is the basic cohomology of the superoperation.

The (basic) cohomology of any $N=2$ superoperation a is structurally restricted, as indicated by the following.

Proposition 4. *One has*

$$H^{n,p}(\mathsf{a}) = 0, \quad \text{for} \;\; p \neq \pm n+1. \tag{65}$$

Similarly,

$$H^{n,p}_{\mathrm{basic}}(\mathsf{a}) = 0, \quad \text{for} \;\; p \neq \pm n+1. \tag{66}$$

Proof. It is convenient for the time being to revert to the original basis $h_A, h_{B,C}, \tilde{h}_D$ of t, which allows for a more compact notation. Let $x\in\mathsf{a}$ such that $\tilde{h}_A x = 0$. Using (35b,c), it is easy to show that

$$[h_A + \epsilon^{KL}h_K h_{L,A}]x - \tilde{h}_A \tfrac{1}{2}\epsilon^{KL}h_K h_L x = 0. \tag{67}$$

Apply now $\tilde{h}_B$ to the left-hand side of this equation and contract with ϵ^{BA}. After a short calculation exploiting (35c,e), one gets

$$[-\tfrac{1}{2}\epsilon^{KL}\epsilon^{MN}h_{K,M}h_{L,N} + \tfrac{1}{2}\epsilon^{KL}h_{K,L}]x - \tfrac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L\tfrac{1}{2}\epsilon^{MN}h_M h_N x = 0. \tag{68}$$

Using the relation $h_{A,B} = \frac{1}{2}(t_{AB} - \epsilon_{AB}k)$, following from (38a,b) and (61) in (68), one gets finally

$$[c + \tfrac{1}{4}(1-(k-1)^2)]x - \tfrac{1}{2}\epsilon^{KL}d_K d_L \tfrac{1}{2}\epsilon^{MN}u_M u_N x = 0. \tag{69}$$

If $x \in \mathsf{a}^{n,p}$, (69) yields

$$\tfrac{1}{4}[n^2 - (p-1)^2]x - \tfrac{1}{2}\epsilon^{KL}d_K d_L \tfrac{1}{2}\epsilon^{MN}u_M u_N x = 0. \tag{70}$$

(70) yields (65) immediately. (66) follows also from (70) upon checking that for $x \in \mathsf{a}_{\text{basic}}$, $\frac{1}{2}\epsilon^{MN}h_M h_N x \in \mathsf{a}_{\text{basic}}$ as well, by (37a–c). □

Proposition 5. *The nontrivial elements of* $H^{n,p}(\mathsf{a})$ $(H^{n,p}_{\text{basic}}(\mathsf{a}))$ *fill irreducible representations of the internal symmetry algebra* i *of invariants* $n,\ p$.

Proof. By (41c,d), if $x \in \mathsf{a}^{n,p} \cap \cap_{A=1,2}\ker d_A$, then $t_{AB}x, kx \in \mathsf{a}^{n,p} \cap \cap_{A=1,2}\ker d_A$ as well. Further, if $x \in \frac{1}{2}\epsilon^{KL}d_K d_L \mathsf{a}^{n,p-2}$, $t_{AB}x,\ \ kx \in \frac{1}{2}\epsilon^{KL}d_K d_L \mathsf{a}^{n,p-2}$, also. One thus defines $t_{AB}[x] = [t_{AB}x]$ and $k[x] = [kx]$, for any $[x] \in H^{n,p}(\mathsf{a})$ with arbitrary representative $x \in \mathsf{a}^{n,p} \cap \cap_{A=1,2}\ker d_A$. This yields the first part of the proposition. The statement extends to basic cohomology, by noting that $t_{AB}x,\ \ kx \in \mathsf{a}^{n,p}_{\text{basic}}$ whenever $x \in \mathsf{a}^{n,p}_{\text{basic}}$, by (41e–j). □

Recall that the only irreducible n dimensional module of $\mathsf{i} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathbb{R}$ is the completely symmetric tensor space $\bigvee^{n-1}\mathbb{R}^2$ up to equivalence. Hence, one has a tensor factorization of the form

$$H^{n,p}(\mathsf{a}) = K^{n,p} \otimes \bigvee^{n-1}\mathbb{R}^2, \tag{71}$$

$$H^{n,p}_{\text{basic}}(\mathsf{a}) = K^{n,p}_{\text{basic}} \otimes \bigvee^{n-1}\mathbb{R}^2, \tag{72}$$

for certain vector spaces $K^{n,p}$, $K^{n,p}_{\text{basic}}$.

6. The Weil superoperation and its cohomologies

Let $\mathfrak{g}$ be an ungraded Lie algebra.

6.1. The $N = 1$ case

Let w be the $N = 1$ Weil algebra of $\mathfrak{g}$ (cf. Section 4.1). Then, w is an $N = 1$ $\mathfrak{g}$ superoperation (cf. Definition 7) called $N = 1$ Weil superoperation. Indeed, as shown in Section

4.1, W is a $\mathbb{Z}$ graded left module algebra of the fundamental $N = 1$ superstructure t of $\mathfrak{g}$, the action of t on W is derivative and W is obviously completely reducible under the internal symmetry algebra i with k acting as the degree operator of W by (49a,b).

Theorem 1. *$H^p(\mathsf{W}) = 0$ for $p \neq 0$ and*

$$H^0(\mathsf{W}) \simeq \mathbb{R}. \tag{73}$$

Similarly, $H^p_{\mathrm{basic}}(\mathsf{W}) = 0$, for $p \neq 2s$ with $s \geq 0$, and

$$H^{2s}_{\mathrm{basic}}(\mathsf{W}) \simeq \left(\bigvee^s \mathfrak{g}^\vee\right)_{\mathrm{ad}^\vee \mathfrak{g}}, \quad s \geq 0, \tag{74}$$

where $(\bigvee^s \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}}$ denotes the subspace of symmetrized tensor product $\bigvee^s \mathfrak{g}^\vee$ spanned by the elements which are invariant under the coadjoint action of $\mathfrak{g}$.

Proof. Below, we shall use the following notation. Let $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee$. Let $\xi \in \Pi\mathfrak{g}$, $\eta \in \mathfrak{g}$, where $\Pi\mathfrak{g}$ is the Grassmann odd partner of $\mathfrak{g}$. We denote by $r(\xi, \eta)$ the evaluation of r on $\sum_{p,q\geq 0} \xi^{\otimes p} \otimes \eta^{\otimes q}$. Every element $z \in \mathsf{W}$ is of the form $z = r(w, \tilde{w})$ for some $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee$ uniquely determined by z. As $\deg w = 1$, $\deg \tilde{w} = 2$, $\mathsf{W}^p = 0$ for $p < 0$ and $\mathsf{W}^0 = \mathbb{R}1$. Hence, $H^p(\mathsf{W}) = 0$ for $p < 0$ and $H^0(\mathsf{W}) \simeq \mathbb{R}$, trivially. Let $\mathsf{W}^{p>0} = \oplus_{p>0} \mathsf{W}^p$. $\mathsf{W}^{p>0}$ is acted upon by the graded derivations $h, \tilde{h}$ and two more graded derivations $i^*, \tilde{i}^*$ of degree $-1, 0$, respectively, defined by

$$i^* w = 0, \quad i^* \tilde{w} = w, \qquad \tilde{i}^* w = w, \quad \tilde{i}^* \tilde{w} = \tilde{w}. \tag{75a–d}$$

Identify $i^*, \tilde{i}^*$ with the linear maps $i^*(x) = x i^*$, $\tilde{i}^*(x) = x\tilde{i}^*$, $x \in \mathbb{R}$. Then, $h, \tilde{h}, i^*, \tilde{i}^*$ satisfy relations (24)–(26) with $\mathfrak{g} = \mathbb{R}$. It follows that $\mathsf{W}^{p>0}$ is an $N = 1\,\mathbb{R}$ superoperation. Switch now to the derivations k, d, j^*, l^* defined by (27) and (28). By (32a), j^* is a homotopy operator for d, for l^* commutes with j^* and d, by (32b) and (33b), and l^* is invertible on $\mathsf{W}^{p>0}$, by (75c,d) and the definition of $\mathsf{W}^{p>0}$. Thus, the cohomology of d is trivial on $\mathsf{W}^{p>0}$. This proves the first part of the theorem. Every element $z \in \mathsf{W}_{\mathrm{basic}}$ is of the form $z = r(\phi)$ for some $r \in (\bigvee^* \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}}$ uniquely determined by z. Indeed, $z = r(\omega, \phi)$ for a unique $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee$, by an argument similar to that employed earlier, and, by (51), the basicity conditions $j(\xi)r(\omega, \phi) = 0$, $l(\xi)r(\omega, \phi) = 0$ imply that r has polynomial degree 0 in the first argument and is $\mathrm{ad}^\vee \mathfrak{g}$ invariant. It follows that $\mathsf{W}^p_{\mathrm{basic}} = 0$ for $p \neq 2s$ with $s \geq 0$, as $\deg \phi = 2$. So, $H^p_{\mathrm{basic}}(\mathsf{W}) = 0$ for $p \neq 2s$ with $s \geq 0$. Let $s \geq 0$. If $z = r(\phi)$ with $r \in (\bigvee^s \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}}$, then $\mathrm{d}z = 0$, by (50b) and the $\mathrm{ad}^\vee \mathfrak{g}$ invariance of r. Hence, $\mathsf{W}^{2s}_{\mathrm{basic}} \cap \ker d = \mathsf{W}^{2s}_{\mathrm{basic}}$. We thus have a linear injection $\mu : \mathsf{W}^{2s}_{\mathrm{basic}} \cap \ker d \mapsto (\bigvee^s \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}}$ given by $z \mapsto r$. As, $\mathsf{W}^{2s-1}_{\mathrm{basic}} = 0$, μ induces a linear bijection $\hat{\mu} : H^{2s}_{\mathrm{basic}}(\mathsf{W}) \mapsto (\bigvee^s \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}}$. $\square$

6.2. The $N = 2$ case

Let W be the $N = 2$ Weil algebra of $\mathfrak{g}$ (cf. Section 4.2). Then, W is an $N = 2\,\mathfrak{g}$ superoperation (cf. Definition 8) called $N = 2$ Weil superoperation. Indeed, as shown in Section

4.2, w is a $\mathbb{Z}$ graded left module algebra of the fundamental $N = 2$ superstructure t of $\mathfrak{g}$, the action of t on w is derivative and w is obviously completely reducible under the internal symmetry algebra i with k acting as the degree operator of w by (55).

Theorem 2. *$H^{n,p}(\mathsf{w}) = 0$, for $(n, p) \neq (1, 0)$, and*

$$H^{1,0}(\mathsf{w}) \simeq \mathbb{R}. \tag{76}$$

Similarly, $H^{n,p}_{\mathrm{basic}}(\mathsf{w}) = 0$, for $(n, p) \neq (1, 0)$, $(2s, 2s + 1)$ with $s > 0$, and

$$H^{1,0}_{\mathrm{basic}}(\mathsf{w}) \simeq \mathbb{R}, \qquad H^{2s,2s+1}_{\mathrm{basic}}(\mathsf{w}) \simeq \left(\bigvee^{s} \mathfrak{g}^\vee\right)_{\mathrm{ad}^\vee\mathfrak{g}} \otimes \bigvee^{2s-1} \mathbb{R}^2, \quad s > 0. \tag{77}$$

Proof. Below, we shall use the following notation. Let $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \bigotimes^a \mathbb{R}^2) \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \bigotimes^b \mathbb{R}^2)$. Let $\xi \in \Pi\mathfrak{g} \otimes \bigotimes^a \mathbb{R}^{2\vee}$, $\eta \in \mathfrak{g} \bigotimes^b \mathbb{R}^{2\vee}$. We denote by $r(\xi, \eta)$ the evaluation of r on $\sum_{p,q\geq 0} \xi^{\otimes p} \otimes \eta^{\otimes q}$. The above notation can be straightforwardly generalized to the case where there are several ξ and η. Every element $z \in \mathsf{w}$ is of the form $z = r(w, w_{,}, \tilde{w})$ for some $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \bigotimes^2 \mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2)$ uniquely determined by z. As $\deg w_A = 1$, $\deg w_{A,B} = 2$, $\deg \tilde{w}_A = 3$, $\mathsf{w}^{n,p} = 0$, for $p < 0$, and $\mathsf{w}^{n,0} = \mathbb{R}\delta_{n,1}1$. So, $H^{n,p}(\mathsf{w}) = 0$, for $p < 0$, and $H^{n,0}(\mathsf{w}) \simeq \delta_{n,1}\mathbb{R}$, trivially. Let $\mathsf{w}^{p>0} = \bigoplus_{n\in\mathbb{N},p>0} \mathsf{w}^{n,p}$. $\mathsf{w}^{p>0}$ is acted upon by the graded derivations $h_A, h_{A,B}, \tilde{h}_A$ and three more graded derivations $i, i_{,A}, \tilde{i}$ of degree $-2, -1, 0$, respectively, defined by

$$\begin{aligned} &i^* w_A = 0, && i^* w_{A,B} = 0, && i^* \tilde{w}_A = w_A, \\ &i^*_{,A} w_B = 0, && i^*_{,A} w_{B,C} = -\epsilon_{CA} w_B, && i^*_{,A} \tilde{w}_B = w_{B,A}, \\ &\tilde{i}^* w_A = w_A, && \tilde{i}^* w_{A,B} = w_{A,B}, && \tilde{i}^* \tilde{w}_A = \tilde{w}_A. \end{aligned} \tag{78a–i}$$

Identify $i^*, i^*_{,A}, \tilde{i}^*$ with the linear maps $i^*(x) = xi^*$, $i^*_{,A}(x) = xi^*_{,A}$, $\tilde{i}^*(x) = x\tilde{i}^*$, $x \in \mathbb{R}$. Then, $h_A, h_{A,B}, \tilde{h}_A, i, i_{,A}, \tilde{i}$ satisfy relations (35)–(37) with $\mathfrak{g} = \mathbb{R}$. From this fact, it is easy to see that $\mathsf{w}^{p>0}$ is an $N = 2\,\mathbb{R}$ superoperation. Switch now to the derivations $t_{A,B}, k, u_A, d_A, j^*, j^*_A, l^*$ defined by (38) and (39). By (43e), j^*_A is a homotopy operator for d_A, for l^* commutes with j^*_A and d_A, by (43f) and (44e), and l^* is invertible on $\mathsf{w}^{p>0}$, by (78g–i) and the definition of $\mathsf{w}^{p>0}$. Indeed, using (43e,f) and (44e), one can show that

$$[\tfrac{1}{2}\epsilon^{KL} d_K d_L, \tfrac{1}{2}\epsilon^{MN} j^*_M j^*_N] = -l^*(l^* + \epsilon^{KL} j^*_K d_L), \tag{79}$$

where, by (41g,h), $\frac{1}{2}\epsilon^{MN} j^*_M j^*_N$ maps $\mathsf{w}^{n,p}$ into $\mathsf{w}^{n,p-2}$. Thus, the cohomology of d_A is trivial on $\mathsf{w}^{p>0}$. This proves the first part of the theorem. Let us examine next the second part. As $\mathsf{w}^{n,p} = 0$, for $p < 0$, and $\mathsf{w}^{n,0} = \mathbb{R}\delta_{n,1}1$, as shown earlier, and 1 is obviously basic, $\mathsf{w}^{n,p}_{\mathrm{basic}} = 0$, for $p < 0$, and $\mathsf{w}^{n,0}_{\mathrm{basic}} = \mathbb{R}\delta_{n,1}1$. Consequently, $H^{n,p}_{\mathrm{basic}}(\mathsf{w}) = 0$ for $p < 0$. and $H^{n,0}_{\mathrm{basic}}(\mathsf{w}) \simeq \delta_{n,1}\mathbb{R}$. On the other hand, by Proposition 4, Eq. (66), $H^{n,p}_{\mathrm{basic}}(\mathsf{w}) = 0$ for $p \neq \pm n + 1$. So, the only potentially nonvanishing cohomology spaces which are left are $H^{n,n+1}_{\mathrm{basic}}(\mathsf{w})$, $n \geq 1$, which we shall analyze next. Every element $z \in \mathsf{w}_{\mathrm{basic}}$ is of the form $z = r(\phi, \rho)$ for some $r \in (\bigvee^*(\mathfrak{g}^\vee \otimes \bigvee^2 \mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2))_{\mathrm{ad}^\vee\mathfrak{g}}$ uniquely determined by z. Indeed, $z = r(\omega, \gamma, \phi, \rho)$, for a unique $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \bigvee^* \mathfrak{g}^\vee \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \bigvee^2 \mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2)$, by an argument similar to that employed earlier in the proof, and, by (57), the basicity conditions $j(\xi)r(\omega,\gamma,\phi,\rho)=0$, $j_A(\xi)r(\omega,\gamma,\phi,\rho)=0$, $l(\xi)r(\omega,\gamma,\phi,\rho)=0$ imply that r has polynomial degree 0 in the first two arguments and is $\mathrm{ad}\,\mathfrak{g}$ invariant. Let $z=r(\phi,\rho)\in \mathsf{W}_{\mathrm{basic}}^{n,n+1}$. From (55c,d,g,h) and the representation theory of $\mathsf{i}=\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$, one knows that the total number of internal indices $A=1,2$ and the total degree carried by ϕ_{AB}, ρ_A in each monomial of $r(\phi,\rho)$ must be $n-1+2\nu$ and $n+1$, respectively, where 2ν is the number of indices contracted by means of ϵ^{AB}. Further, the $n-1$ uncontracted indices are totally symmetrized. So, the numbers m_ϕ, m_ρ of occurrences of ϕ_{AB}, ρ_A in a given monomial must satisfy the equations $2m_\phi+1m_\rho=n-1+2\nu$, $2m_\phi+3m_\rho=n+1$. Taking into account that m_ϕ, m_ρ are nonnegative integers, one finds that $\nu=0, m_\phi=s-1, m_\rho=1$, for $n=2s$ with $s\geq 1$, and $\nu=1, m_\phi=s, m_\rho=0$, for $n=2s-1$ with $s\geq 2$. Thus, the most general $z\in\mathsf{W}_{\mathrm{basic}}^{n,n+1}$ is of the form

$$z=u^{A_1\cdots A_{2s-1}}(\phi_{A_1A_s},\ldots,\phi_{A_{s-1}A_{2s-2}},\rho_{A_{2s-1}}),\quad n=2s,\quad s\geq 1, \tag{80}$$

$$z=\tfrac{1}{2}\epsilon^{KL}v^{A_1\cdots A_{2s-2}}(\phi_{A_1A_{s-1}},\ldots,\phi_{A_{s-2}A_{2s-4}}\phi_{A_{2s-3}K}\phi_{A_{2s-2}L}),\quad n=2s-1,\quad s\geq 2, \tag{81}$$

where $u^{A_1\cdots A_{2s-1}}\in(\bigvee^{s-1}\mathfrak{g}^\vee\otimes\mathfrak{g}^\vee)_{\mathrm{ad}^\vee\mathfrak{g}}$ totally symmetric in $A_1,\ldots,A_{2s-1}$, $v^{A_1\cdots A_{2s-2}}\in(\bigvee^{s-2}\mathfrak{g}^\vee\otimes\bigwedge^2\mathfrak{g}^\vee)_{\mathrm{ad}^\vee\mathfrak{g}}$ totally symmetric in $A_1,\ldots,A_{2s-2}$. Suppose now that $z\in\mathsf{W}_{\mathrm{basic}}^{n,n+1}\cap\bigcap_{A=1,2}\ker d_A$ so that z, besides being of the form (80) and (81), satisfies $d_Az=0$. Suppose first that $n=2s$. Using (80) and (56f,h), the symmetry properties and the $\mathrm{ad}\,\mathfrak{g}$ invariance of $u^{A_1\cdots A_{2s-1}}$ and taking into account that terms with a different number of occurrences of ϕ_{AB}, ρ_A are linearly independent, the condition $d_Az=0$ is equivalent to the equations

$$\epsilon_{AA_{s-1}}u^{A_1\cdots A_{2s-1}}(\phi_{A_1A_s},\ldots,\phi_{A_{s-2}A_{2s-3}},\rho_{A_{2s-2}},\rho_{A_{2s-1}})=0, \tag{82}$$

$$\tfrac{1}{2}\epsilon^{KL}u^{A_1\cdots A_{2s-1}}(\phi_{A_1A_s},\ldots,\phi_{A_{s-1}A_{2s-2}},[\phi_{KA_{2s-1}},\phi_{LA}])=0. \tag{83}$$

As ρ_A is odd and $u^{A_1\cdots A_{2s-1}}$ is totally symmetric in $A_1,\ldots,A_{2s-1}$, (82) entails that $u^{A_1\cdots A_{2s-1}}$ is totally symmetric in its s arguments. Using this fact and the $\mathrm{ad}\,\mathfrak{g}$ invariance of $u^{A_1\cdots A_{2s-1}}$, it is easy to see that (83) is identically satisfied. Hence, $u^{A_1\cdots A_{2s-1}}\in(\bigvee^s\mathfrak{g}^\vee)_{\mathrm{ad}^\vee\mathfrak{g}}$. Conversely, if this holds, then (82) and (83) are fulfilled. The above analysis shows that $\mathsf{W}_{\mathrm{basic}}^{2s,2s+1}\cap\bigcap_{A=1,2}\ker d_A$ is precisely the space of the z of the form (80) with $u^{A_1\cdots A_{2s-1}}\in(\bigvee^s\mathfrak{g}^\vee)_{\mathrm{ad}^\vee\mathfrak{g}}$ totally symmetric in $A_1,\ldots,A_{2s-1}$. Thus, we have a linear bijection $\mu:\mathsf{W}_{\mathrm{basic}}^{2s,2s+1}\cap\bigcap_{A=1,2}\ker d_A\mapsto(\bigvee^s\mathfrak{g}^\vee)_{\mathrm{ad}^\vee\mathfrak{g}}\otimes\bigvee^{2s-1}\mathbb{R}^2$ defined by $z\mapsto(u^{A_1\cdots A_{2s-1}})_{A_1,\ldots,A_{2s-1}=1,2}$. We note next that $\mathsf{W}_{\mathrm{basic}}^{2s,2s-1}=0$. Indeed, if $z=r(\phi,\rho)\in\mathsf{W}_{\mathrm{basic}}^{n,n-1}$, the total number of internal indices $A=1,2$ and the total degree carried by ϕ_{AB}, ρ_A in each monomial of $r(\phi,\rho)$ must be $n-1+2\nu$ and $n-1$, respectively, where 2ν is the number of indices contracted by means of ϵ^{AB}. So, the numbers m_ϕ, m_ρ of occurrences of ϕ_{AB}, ρ_A in a given monomial must satisfy the equations $2m_\phi+1m_\rho=n-1+2\nu$, $2m_\phi+3m_\rho=n-1$. Taking into account that m_ϕ, m_ρ are nonnegative integers, one finds that there are no solutions for $n=2s$ with $s>0$, so that $\mathsf{W}_{\mathrm{basic}}^{2s,2s-1}=0$ as announced. Thus, the bijection μ above induces a bijection $\hat\mu: H_{\mathrm{basic}}^{2s,2s+1}(\mathsf{W})\mapsto$

$(\bigvee^s \mathfrak{g}^\vee)_{\mathrm{ad}^\vee \mathfrak{g}} \otimes \bigvee^{2s-1} \mathbb{R}^2$. Suppose next that $n = 2s-1$. Using (81) and (56f,h), the (anti)symmetry properties and the ad$\mathfrak{g}$ invariance of $v^{A_1\cdots A_{2s-2}}$, the condition $d_A z = 0$ is equivalent to the equation

$$\begin{aligned}&(s-2)\epsilon^{KL}\epsilon_{AA_{s-2}} v^{A_1\cdots A_{2s-2}}(\phi_{A_1A_{s-1}}, \ldots, \phi_{A_{s-3}A_{2s-5}}, \rho_{A_{2s-4}}, \phi_{A_{2s-3}K}, \phi_{A_{2s-2}L})\\ &\quad + v^{A_1\cdots A_{2s-2}}(\phi_{A_1A_{s-1}}, \ldots, \phi_{A_{s-2}A_{2s-4}}, \rho_{A_{2s-3}}, \phi_{A_{2s-2}A})\\ &\quad + \epsilon^{KL}\epsilon_{AA_{2s-3}} v^{A_1\cdots A_{2s-2}}(\phi_{A_1A_{s-1}}, \ldots, \phi_{A_{s-2}A_{2s-4}}, \rho_K, \phi_{A_{2s-2}L}) = 0. \qquad (84)\end{aligned}$$

Now, apply u_B to this relation, using (56b,d), and then contract with ϵ^{BA}. One gets then $\frac{1}{2}\epsilon^{KL} v^{A_1\cdots A_{2s-2}}(\phi_{A_1A_{s-1}}, \ldots, \phi_{A_{s-2}A_{2s-4}}\phi_{A_{2s-3}K}\phi_{A_{2s-2}L}) = 0$. So, $z = 0$. We conclude that $\mathsf{w}_{\mathrm{basic}}^{2s-1,2s} \cap \bigcap_{A=1,2} \ker d_A = 0$. Thus, $H_{\mathrm{basic}}^{2s-1,2s}(\mathsf{w}) = 0$ as well. □

6.3. *The relation between the cohomologies of the $N = 1$ and $N = 2$ Weil superoperations*

Let $\mathsf{w}(n)$ denote the $N = n$ Weil superoperations, $n = 1, 2$.

Corollary 1. *One has*

$$H^{n,\pm n+1}(\mathsf{w}(2)) \simeq H^{\pm(n-1/2)+1/2}(\mathsf{w}(1)) \otimes \bigvee^{n-1} \mathbb{R}^2, \qquad (85)$$

$$H_{\mathrm{basic}}^{n,\pm n+1}(\mathsf{w}(2)) \simeq H_{\mathrm{basic}}^{\pm(n-1/2)+1/2}(\mathsf{w}(1)) \otimes \bigvee^{n-1} \mathbb{R}^2. \qquad (86)$$

Proof. Combine Theorems 1 and 2. □

Thus, the $N = 1$ and $N = 2$ cohomologies of w are intimately related.

7. Connections, equivariant cohomology and Weil homomorphism

Let $\mathfrak{g}$ be an ungraded Lie algebra.

7.1. *The $N = 1$ case*

Let a be an $N = 1$ $\mathfrak{g}$ superoperation with unity, i.e. a as an algebra has a unity 1.

Definition 9. A connection a on a is an element of $\mathsf{a} \otimes \mathfrak{g}$ satisfying relations (49a) and (51a,c) with ω substituted by a.

The curvature of a is defined as usual as

$$f = da + \tfrac{1}{2}[a, a]. \qquad (87)$$

It is easy to see that f satisfies relations (49b) and (51b,d) with ϕ substituted by f. In particular, being $j(\xi)f = 0$ for any $\xi \in \mathfrak{g}$, f is horizontal. a, f together fulfil (50).

We denote by $\mathrm{Conn}(\mathsf{a})$ the set of the connections of the $N = 1\mathfrak{g}$ superoperation a. $\mathrm{Conn}(\mathsf{a})$ is an affine space modelled on $\mathsf{a}^1 \otimes \mathfrak{g}$.

Proposition 6. *Let $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee$ be such that, for any connection $a \in \mathrm{Conn}(\mathsf{a})$, $r(a, f)$ is a representative of some element of $H^p_{\mathrm{basic}}(\mathsf{a})$ (see above Eq. (75) for the definition of the notation). Then, the basic cohomology class $[r(a, f)]$ is independent from the choice of a.*

Proof. We follow the methods of Ref. [32]. Consider the $N = 1$ superoperation s generated by s, $\tilde{s}$ of degree $0, +1$, respectively, with

$$\begin{aligned} &h^s s = 0, \qquad h^s \tilde{s} = \tilde{s}, \\ &\tilde{h}^s s = -\tilde{s}, \quad \tilde{h}^s \tilde{s} = 0, \end{aligned} \tag{88a–d}$$

$$i^s(\xi) = 0, \quad \tilde{i}^s(\xi) = 0, \qquad \xi \in \mathfrak{g}. \tag{89a,b}$$

Next, we consider the graded tensor product superoperation $\mathsf{s}\hat{\otimes}\mathsf{a}$ and the subalgebra c of $\mathsf{s}\hat{\otimes}\mathsf{a}$ generated by the elements of the form $a(s)$, $\tilde{h}^s a(s)$, $\tilde{a}(s)$, $\tilde{h}^s \tilde{a}(s)$, where $a : \mathbb{R} \mapsto \mathsf{a}\otimes\mathfrak{g}$ is a polynomial such that, for fixed $\sigma \in \mathbb{R}$, $a(\sigma)$ is a connection on a and $\tilde{a}(\sigma) = -\tilde{h}a(\sigma)$. Next, we define a degree 0 derivation q on c by

$$\begin{aligned} &qa(s) = 0, \qquad q\tilde{a}(s) = -\tilde{h}^s a(s), \\ &q\tilde{h}^s a(s) = 0, \quad q\tilde{h}^s \tilde{a}(s) = 0. \end{aligned} \tag{90a–d}$$

Note that, for fixed $\sigma \in \mathbb{R}$, $a(\sigma)$, $\tilde{a}(\sigma)$ satisfy relations (46) and (47) with w, $\tilde{w}$ replaced by $a(\sigma)$, $\tilde{a}(\sigma)$. Using this fact, one easily checks that

$$[q, \tilde{h}] = \tilde{h}^s, \quad [q, \tilde{h}^s] = 0, \tag{91a,b}$$

$$[q, h + h^s] = 0, \tag{92}$$

$$[q, i(\xi)] = 0, \quad [q, \tilde{i}(\xi)] = 0, \qquad \xi \in \mathfrak{g}. \tag{93a,b}$$

Let $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee$ be such that, for any connection a on a, $r[a] := r(a, \tilde{a})$ belongs to $\mathsf{a}_{\mathrm{basic}} \cap \ker \tilde{h}$. By (91a) and the fact that $\tilde{h}r[a] = 0$,

$$\tilde{h}^s r[a(s)] = -\tilde{h}qr[a(s)]. \tag{94}$$

We note that, by (88c,d) and (90a,b), $qr[a(s)]$ is necessarily of the form $qr[a(s)] = \tilde{s}\alpha(s|a)$, where $\alpha(s|a)$ is a polynomial in s. From this expression and (88a,b), it follows that $h^s qr[a(s)] = qr[a(s)]$. By (92), one has then

$$hqr[a(s)] = q(h - 1)r[a(s)]. \tag{95}$$

Further, from (93) and the fact that $i(\xi)r[a] = 0$, $\tilde{i}(\xi)r[a] = 0$,

$$i(\xi)qr[a(s)] = 0, \quad \tilde{i}(\xi)qr[a(s)] = 0, \qquad \xi \in \mathfrak{g}. \tag{96a,b}$$

For any element x of $\mathsf{s}\hat{\otimes}\mathsf{a}$ of the form $x = \tilde{s}\alpha(s)$ with $\alpha(s)$ a polynomial in s, we define $\int_{[0,1]} x = \int_0^1 \alpha(\sigma)\,\mathrm{d}\sigma$, where the right-hand side is an ordinary Riemann integral. It is obvious that, for any element of $f(s)$ of $\mathsf{s}\hat{\otimes}\mathsf{a}$ polynomial in s, $\tilde{h}^s f(s)$ is of the above form and $-\int_{[0,1]} \tilde{h}^s f(s) = f(1) - f(0)$. From (94), one has thus

$$r[a(1)] - r[a(0)] = \tilde{h} \int_{[0,1]} qr[a(s)]. \tag{97}$$

By (27b), the right-hand side of (97) belongs to $d\mathsf{a}$. From (27a), (95) and (96), if $r[a]$ belongs to $\mathsf{a}_{\mathrm{basic}}^{p}$ for any connection a on a, then $qr[a(\sigma)]$ belongs to $\mathsf{a}_{\mathrm{basic}}^{p-1}$ for $\sigma \in \mathbb{R}$, so that $\int_{[0,1]} qr[a(s)]$ belongs to $\mathsf{a}_{\mathrm{basic}}^{p-1}$, too. □

Consider the $N = 1$ Weil $\mathfrak{g}$ superoperation w (cf. Section 6.1). Then ω is a connection on w with curvature ϕ (cf. Eq. (48)).

Given an $N = 1$ $\mathfrak{g}$ superoperation a with unity, one can define the graded tensor product $N = 1$ $\mathfrak{g}$ superoperation $\mathsf{w}\hat{\otimes}\mathsf{a}$ (cf. Section 5.1). The latter is the equivariant $N = 1$ superoperation associated to a. The equivariant cohomology of a is, by definition the basic cohomology of $\mathsf{w}\hat{\otimes}\mathsf{a}$:

$$H_{\mathrm{equiv}}^{p}(\mathsf{a}) = H_{\mathrm{basic}}^{p}(\mathsf{w}\hat{\otimes}\mathsf{a}), \qquad p \in \mathbb{Z}. \tag{98}$$

An equivariant cohomology class of a is represented by elements of $\mathsf{w}\hat{\otimes}\mathsf{a}$ of the form $r(\omega, \phi)$, where $r \in \bigwedge^* \mathfrak{g}^\vee \otimes \bigvee^* \mathfrak{g}^\vee \otimes \mathsf{a}$. The Weil generator ω constitutes a connection of $\mathsf{w}\hat{\otimes}\mathsf{a}$. If a is a connection of a, a is a connection of $\mathsf{w}\hat{\otimes}\mathsf{a}$ as well. By Proposition 6, $r(\omega, \phi)$ is equivalent to $r(a, f)$ in equivariant cohomology. On the other hand, $r(a, f)$ is a representative of a basic cohomology class of a, which, by Proposition 6, is independent from a in basic cohomology. Thus, there is a natural homomorphism of $H_{\mathrm{equiv}}^{*}(\mathsf{a})$ into $H_{\mathrm{basic}}^{p}(\mathsf{a})$, called $N = 1$ Weil homomorphism.

7.2. *The $N = 2$ case*

Let a be an $N = 2$ $\mathfrak{g}$ superoperation with unity.

Definition 10. A connection $(a_A)_{A=1,2}$, on a is a doublet of $\mathsf{a} \otimes \mathfrak{g}$ satisfying relations (55a,b), (56a), (57a,e,i) with ω_A substituted by a_A.

The derived connection

$$b = \tfrac{1}{2}\epsilon^{KL} d_K a_L \tag{99}$$

and the curvature and derived curvature

$$f_{AB} = \tfrac{1}{2}(d_A a_B + d_B a_A + [a_A, a_B]),$$
$$g_A = -\tfrac{1}{4}\epsilon^{KL} d_K d_L a_A - \tfrac{1}{2}\epsilon^{KL}[a_K, d_L a_A] - \tfrac{1}{6}\epsilon^{KL}[a_K, [a_L, a_A]] \tag{100a,b}$$

satisfy relations (55c–h) and (57b–d,f–h,j–l) with $\gamma, \phi_{AB}, \rho_A$ substituted by b, f_{AB}, g_A, respectively. In particular, being $j(\xi)f_{AB} = 0$, $j_A(\xi)f_{BC} = 0$, $j(\xi)g_A = 0$, $j_A(\xi)g_B = 0$ for any $\xi \in \mathfrak{g}$, f_{AB}, g_A are horizontal. a_A, b, f_{AB}, g_A together satisfy (56b–h).

We denote by Conn(a) the set of connections of the $N = 2$ superoperation a. Conn(a) is an affine space modelled on $\mathsf{a}^{2,1} \otimes \mathfrak{g}$.

Proposition 7. *Let* $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \bigvee^* \mathfrak{g}^\vee \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \bigvee^2 \mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2)$ *be such that, for any connection* $(a_A)_{A=1,2} \in$ Conn(a), $r(a, b, f, g)$ *is a representative of some element of* $H^{n,p}_{\mathrm{basic}}(\mathsf{a})$ *(see above* Eq. (78) *for the definition of the notation). Then, the basic cohomology class* $[r(a, b, f, g)]$ *is independent from the choice of* $(a_A)_{A=1,2}$.

Proof. We generalize the methods of Ref. [32]. Consider the $N = 2$ superoperation s generated by $s, s_{,A}, \tilde{s}$ of degree 0, +1, +2, respectively, with

$$\begin{aligned}
&h^s_A s = 0, && h^s_A s_{,B} = 0,\\
&h^s_A \tilde{s} = -s_{,A}, && h^s_{A,B} s = 0,\\
&h^s_{A,C} s_{,B} = -\epsilon_{BC} s_{,A}, && h^s_{A,B}\tilde{s} = -\epsilon_{AB}\tilde{s},\\
&\tilde{h}^s_A s = -s_{,A}, && \tilde{h}^s_A s_{,B} = \epsilon_{AB}\tilde{s},\\
&\tilde{h}^s_A \tilde{s} = 0,
\end{aligned} \tag{101a–i}$$

$$i^s(\xi) = 0, \quad i^s_{,A}(\xi) = 0, \quad \tilde{i}^s(\xi) = 0, \qquad \xi \in \mathfrak{g}. \tag{102a–c}$$

Next, we consider the graded tensor product superoperation $\mathsf{s}\hat{\otimes}\mathsf{a}$ and the subalgebra c of $\mathsf{s}\hat{\otimes}\mathsf{a}$ generated by the elements of the form $a_A(s), \tilde{h}^s_A a_B(s), \tilde{h}^s_A \tilde{h}^s_B a_C(s)$, $a_{A,B}(s)$, $\tilde{h}^s_A a_{B,C}(s), \tilde{h}^s_A \tilde{h}^s_B a_{C,D}(s), \tilde{a}_A(s), \tilde{h}^s_A \tilde{a}_B(s), \tilde{h}^s_A \tilde{h}^s_B \tilde{a}_C(s)$, where $a_A : \mathbb{R} \mapsto \mathsf{a} \otimes \mathfrak{g}$, $A = 1, 2$, is a polynomial such that, for fixed $\sigma \in \mathbb{R}$, $a_A(\sigma)$ is a connection of a and $a_{A,B}(\sigma) = -\tilde{h}_B a_A(\sigma)$, $\tilde{a}_A(\sigma) = \frac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L a_A(\sigma)$. Next, we define a degree 0 derivation q on c by

$$\begin{aligned}
&q a_A(s) = 0, && q a_{A,B}(s) = -\tilde{h}^s_B a_A(s),\\
&q\tilde{a}_A(s) = -\epsilon^{KL}\tilde{h}^s_K a_{A,L}(s), && q\tilde{h}^s_A a_B(s) = 0,\\
&q\tilde{h}^s_A a_{B,C}(s) = \epsilon_{AC}\tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L a_B(s), && q\tilde{h}^s_A\tilde{a}_B(s) = \tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L a_{B,A}(s),\\
&q\tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L a_A(s) = 0, && q\tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L a_{A,B}(s) = 0,\\
&q\tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L \tilde{a}_A(s) = 0.
\end{aligned} \tag{103a–i}$$

Note that, for fixed $\sigma \in \mathbb{R}$, $a(\sigma), a_A(\sigma), \tilde{a}(\sigma)$ satisfy relations (52) and (53) with $w, w_A, \tilde{w}$ replaced by $a(\sigma), a_A(\sigma), \tilde{a}(\sigma)$. Using this fact, one easily checks that

$$[q, \tilde{h}_A] = \tilde{h}^s_A, \quad [q, \tilde{h}^s_A] = 0, \tag{104a,b}$$

$$[q, h_{A,B} + h^s_{A,B}] = 0, \tag{105}$$

$$[q, i(\xi)] = 0, \quad [q, i_{,A}(\xi)] = 0, \quad [q, \tilde{i}(\xi)] = 0, \qquad \xi \in \mathfrak{g}. \tag{106a–c}$$

Using (104a), it is easy to show that

$$[q, \tfrac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L] = \epsilon^{KL}\tilde{h}^s_K\tilde{h}_L, \quad \tfrac{1}{2}[q, [q, \tfrac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L]] = \tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L. \tag{107a,b}$$

Let $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \otimes^2\mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2)$ be such that, for any connection a_A, $A = 1, 2$, on a, $r[a] := r(a, a_{,}, \tilde{a})$ belongs to $\mathsf{a}_{\mathrm{basic}} \cap \cap_{A=1,2} \ker \tilde{h}_A$. Using (107) and the fact that $\tilde{h}_A r[a] = 0$, it is easy to see that

$$\tfrac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L r[a(s)] = \tfrac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L\tfrac{1}{2}q^2 r[a(s)]. \tag{108}$$

We note that, by (101g–i) and (103a–f), $\frac{1}{2}q^2 r[a(s)]$ is necessarily of the form $\frac{1}{2}q^2 r[a(s)] = \tilde{s}\alpha(s|a) + \frac{1}{2}\epsilon^{KL}s_{,K}s_{,L}\beta(s|a)$, where $\alpha(s|a)$, $\beta(s|a)$ are polynomials in s. From this expression and (101d–f), it follows that $h^s_{A,B}\frac{1}{2}q^2 r[a(s)] = -\epsilon_{AB}\frac{1}{2}q^2 r[a(s)]$. By (105), one has then

$$h_{A,B}\tfrac{1}{2}q^2 r[a(s)] = \tfrac{1}{2}q^2(h_{A,B} + \epsilon_{AB})r[a(s)]. \tag{109}$$

Further, from (106) and the fact that $i(\xi)r[a] = 0$, $i_{,A}(\xi)r[a] = 0$, $\tilde{i}(\xi)r[a] = 0$,

$$i(\xi)\tfrac{1}{2}q^2 r[a(s)] = 0, \quad i_{,A}(\xi)\tfrac{1}{2}q^2 r[a(s)] = 0, \quad \tilde{i}(\xi)\tfrac{1}{2}q^2 r[a(s)] = 0, \qquad \xi \in \mathfrak{g}. \tag{110a–c}$$

For any element x of $\mathsf{s}\hat{\otimes}\mathsf{a}$ of the form $x = \tilde{s}\alpha(s) + \frac{1}{2}\epsilon^{KL}s_{,K}s_{,L}\beta(s)$ with $\alpha(s)$, $\beta(s)$ polynomials in s, we define $\int_{[0,1]} x = \int_0^1 \alpha(\sigma)\,\mathrm{d}\sigma$, where the right-hand side is an ordinary Riemann integral. It is not difficult to show that, for any element of $f(s)$ of $\mathsf{s}\hat{\otimes}\mathsf{a}$ polynomial in s, $\frac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L f(s)$ is of the above form and $\int_{[0,1]}\frac{1}{2}\epsilon^{KL}\tilde{h}^s_K\tilde{h}^s_L f(s) = f(1) - f(0)$. From (108),

$$r[a(1)] - r[a(0)] = \frac{1}{2}\epsilon^{KL}\tilde{h}_K\tilde{h}_L\int_{[0,1]}\frac{1}{2}q^2 r[a(s)]. \tag{111}$$

By (38d), the right-hand side of (111) belongs to $\frac{1}{2}\epsilon^{KL}d_K d_L\mathsf{a}$. From (38a,b), (109) and (110), if $r[a]$ belongs to $\mathsf{a}^{n,p}_{\mathrm{basic}}$ for any connection a_A on a, then $\frac{1}{2}q^2 r[a(\sigma)]$ belongs to $\mathsf{a}^{n,p-2}_{\mathrm{basic}}$ for $\sigma \in \mathbb{R}$, so that $\int_{[0,1]}\frac{1}{2}q^2 r[a(s)]$ belongs to $\mathsf{a}^{n,p-2}_{\mathrm{basic}}$, too. □

Consider the $N = 2$ Weil $\mathfrak{g}$ superoperation w (cf. Section 6.2). Then, ω_A is a connection of w with derived connection γ and curvature and derived curvature ϕ_{AB}, ρ_A (cf. Eq. (54)).

Given an $N = 2$ $\mathfrak{g}$ superoperation a, one can define the graded tensor product $N = 2$ $\mathfrak{g}$ superoperation $\mathsf{w}\hat{\otimes}\mathsf{a}$ (cf. Section 5.2). The latter is the equivariant $N = 2$ superoperation associated to a. The equivariant cohomology of a is by definition the basic cohomology of $\mathsf{w}\hat{\otimes}\mathsf{a}$:

$$H^{n,p}_{\mathrm{equiv}}(\mathsf{a}) = H^{n,p}_{\mathrm{basic}}(\mathsf{w}\hat{\otimes}\mathsf{a}), \qquad (n, p) \in \mathbb{N} \times \mathbb{Z}. \tag{112}$$

An equivariant cohomology class of a is represented by elements of $\mathsf{w}\hat{\otimes}\mathsf{a}$ of the form $r(\omega, \gamma, \phi, \rho)$, where $r \in \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \bigvee^*\mathfrak{g}^\vee \otimes \bigvee^*(\mathfrak{g}^\vee \otimes \bigvee^2\mathbb{R}^2) \otimes \bigwedge^*(\mathfrak{g}^\vee \otimes \mathbb{R}^2) \otimes \mathsf{a}$. The Weil generator ω_A constitutes a connection of $\mathsf{w}\hat{\otimes}\mathsf{a}$. If a_A is a connection of a, a_A is a connection of $\mathsf{w}\hat{\otimes}\mathsf{a}$ as well. By Proposition 7, $r(\omega, \gamma, \phi, \rho)$ is equivalent to $r(a, b, f, g)$ in equivariant cohomology. On the other hand, $r(a, b, f, g)$ is a representative of a basic cohomology class of a, which, by Proposition 7, is independent from a_A in basic cohomology. Thus, there is a natural homomorphism of $H^{n,p}_{\mathrm{equiv}}(\mathsf{a})$ into $H^{n,p}_{\mathrm{basic}}(\mathsf{a})$, called $N = 2$ Weil homomorphism.

8. Superoperations of a smooth manifold with a group action

Let M be a smooth m dimensional manifold. Thus, M is endowed with a collection of smooth charts (U_a, x_a), $a \in A$, in the usual way. Let M carry the right action of a Lie group G with Lie algebra $\mathfrak{g}$ (see Ref. [33] for an exhaustive treatment of the theory of manifolds with a group action).

Let s be a Grassmann algebra such that $\mathsf{s}^0 \simeq \mathbb{R}$.

8.1. $N = 1$ differential geometry

Definition 11. An $N = 1$ differential structure on M is a collection $\{(U_a, X_a)|a \in A\}$, where

1. $\{U_a|a \in A\}$ is an open covering of M;
2. for each $a \in A$, $X_a : U_a \mapsto (\mathsf{S}^0_1)^m$ and $x_a = X_a|_{\theta=0} : U_a \mapsto \mathbb{R}^m$ is a coordinate of M;
3. for $a, b \in A$ such that $U_a \cap U_b \neq \emptyset$, $X_a = x_a \circ x_b^{-1}(X_b)$.

Below, we shall omit the chart indices $a, b, \dots$ except when dealing with matching relations.

We write as usual

$$X^i = x^i + \theta\tilde{x}^i, \quad \tilde{X}^i = \tilde{x}^i, \tag{113a,b}$$

where $x^i : U \mapsto \mathbb{R}$, $\tilde{x}^i : U \mapsto \mathsf{s}^1$.

We introduce the $N = 1$ covariant superderivatives

$$D_i = \tilde{\partial}_{xi} + \theta\partial_{xi}, \quad \tilde{D}_i = \partial_{xi}, \tag{114a,b}$$

where $\tilde{\partial}_{xi} = \partial/\partial\tilde{x}^i$. One has relations

$$[D_i, D_j] = 0, \quad [D_i, \tilde{D}_j] = 0, \quad [\tilde{D}_i, \tilde{D}_j] = 0. \tag{115a–c}$$

Further,

$$D_i X^j = 0, \quad D_i\tilde{X}^j = \delta_i^j, \qquad \tilde{D}_i X^j = \delta_i^j, \quad \tilde{D}_i\tilde{X}^j = 0. \tag{116a–d}$$

Using (113), it is straightforward to check that relations (116) completely characterize $D_i, \tilde{D}_i$.

The transformation properties of X^i under chart changes, stated in Definition 11, imply that

$$\tilde{X}^i_a = \tilde{X}^j_b\tilde{D}_{bj}X^i_a. \tag{117}$$

Using that (116) completely characterize $D_i, \tilde{D}_i$, one can show easily that they match as

$$D_{ai} = \tilde{D}_{ai}X^j_b D_{bj}, \quad \tilde{D}_{ai} = \tilde{D}_{ai}\tilde{X}^j_b D_{bj} + \tilde{D}_{ai}X^j_b\tilde{D}_{bj}. \tag{118a,b}$$

We denote by $\mathcal{F}$ the sheaf of germs of smooth $N = 1$ functions on M generated by $X^i, \tilde{X}^i$. By definition, a generic element $F \in \mathcal{F}(U)$ is a finite sum of the form $F = \sum_{p\geq 0}\phi_{i_1\cdots i_p} \circ X\tilde{X}^{i_1}\cdots\tilde{X}^{i_p}$ for certain smooth maps $\phi_{i_1\cdots i_p} : \mathbb{R}^m \mapsto \mathbb{R}$ antisymmetric in $i_1, \ldots, i_p$. It is easy to see that

$$F = \sum_{p=0}^{m}[F_{i_1\cdots i_p} + \theta\partial_{xi_0}F_{i_i\cdots i_p}\tilde{x}^{i_0}]\tilde{x}^{i_1}\cdots\tilde{x}^{i_p}, \tag{119}$$

where $F_{i_1\cdots i_p} = \phi_{i_1\cdots i_p} \circ x$. Hence, F is completely determined by $f = F|_{\theta=0}$.

$\mathcal{F}$ has a natural grading corresponding to the total s degree of $\tilde{x}^i$.

We define on $U_a \cap U_b \neq \emptyset$,

$$Z_{ab}{}^i{}_j = \tilde{D}_{bj}X_a^i. \tag{120}$$

It is easy to see that Z is a $\mathrm{GL}(m, \mathcal{F})$ 1-cocycle on M. Z is called the fundamental 1-cocycle of the $N = 1$ differential structure. One can introduce in standard fashion the sheaf $\mathcal{F}_{r,s} := \mathcal{F}(Z^{\otimes r} \otimes Z^{\vee\otimes s})$ of germs of smooth $N = 1$ sections of $Z^{\otimes r} \otimes Z^{\vee\otimes s}$. We denote by $\mathsf{f}_{r,s}$ the vector space of sections of $\mathcal{F}_{r,s}$ on M.

$z = Z|_{\theta=0}$ is nothing but the tangent bundle 1-cocycle of M. By (113b), (117) and (119), $\mathsf{f}^p_{r,s}$ can be identified with the space of smooth type r, s tensor valued differential p-forms on M.

We are particularly interested in the space $\mathsf{f}_{0,0}$, which is a graded algebra.

We define

$$H = \tilde{X}^i D_i, \qquad \tilde{H} = -\tilde{X}^i\tilde{D}_i. \tag{121a,b}$$

Using (117) and (118), it is easy to see that $H, \tilde{H}$ are globally defined derivations on $\mathsf{f}_{0,0}$.

Denoting by $c\xi$ the fundamental vector field on M induced by $\xi \in \mathfrak{g}$, we define further

$$I(\xi) = C^i\xi D_i, \qquad \tilde{I}(\xi) = C^i\xi\tilde{D}_i + \tilde{X}^j\tilde{D}_jC^i\xi D_i, \tag{122a,b}$$

where $C\xi$ is the element of $\mathsf{f}^0_{1,0}$ corresponding to $c\xi$ given explicitly by $C^i\xi = c^i\xi + \theta\tilde{x}^j\partial_{xj}c^i\xi$. By (118a) and (120), $I(\xi), \tilde{I}(\xi)$ are also globally defined derivations on $\mathsf{f}_{0,0}$.

Using the relation $D_iC^j\xi = 0$, it is now straightforward to verify that $H, \tilde{H}, I, \tilde{I}$ satisfy relations (24)–(26). In this way, $\mathsf{f}_{0,0}$ becomes a $\mathbb{Z}$ graded left module algebra of the $\mathbb{Z}$ graded Lie algebra $\mathfrak{t}$ (cf. Section 3.1).

Thus, $\mathsf{f} := \mathsf{f}_{0,0}$ acquires the structure of $N = 1\,\mathfrak{g}$ superoperation (cf. Definition 7), the relevant graded derivations being

$$h = \tilde{x}^i\tilde{\partial}_{xi}, \quad \tilde{h} = -\tilde{x}^i\partial_{xi}, \tag{123a,b}$$

$$i(\xi) = c^i\xi\tilde{\partial}_{xi}, \quad \tilde{i}(\xi) = c^i\xi\partial_{xi} + \tilde{x}^j\partial_{xj}c^i\xi\tilde{\partial}_{xi}. \tag{124a,b}$$

This superoperation is canonically associated to the $N = 1$ differential structure.

Now, from (117) and (119), it appears that the graded algebra f is isomorphic to the graded algebra of ordinary differential forms on M. Under such an isomorphism, the derivations $k, d, j(\xi), l(\xi)$, defined in (27) and (28), correspond to the form degree k_{dR}, the de Rham differential d_{dR}, the contraction $j_{\mathrm{dR}}(\xi)$ and the Lie derivative $l_{\mathrm{dR}}(\xi)$, respectively. Therefore, the above is nothing but a reformulation of the customary theory of differential forms, so that, in particular, the (basic) cohomology of f is isomorphic to the (basic) de Rham cohomology.

Theorem 3. *There is an isomorphism of the $N = 1$ (basic) cohomology of f the de Rham (basic) cohomology of the (G) manifold M. Indeed, one has that $H^p(\mathsf{f}) = 0$ ($H^p_{\mathrm{basic}}(\mathsf{f}) = 0$), except perhaps for $0 \leq p \leq m$, and*

$$H^p(\mathsf{f}) \simeq H^p_{\mathrm{dR}}(M), \quad 0 \leq p \leq m, \tag{125}$$

$$H^p_{\mathrm{basic}}(\mathsf{f}) \simeq H^p_{\mathrm{dR\ basic}}(M), \quad 0 \leq p \leq m. \tag{126}$$

Proof. See the above remarks. □

Recall that a connection y on the G space M is a $\mathfrak{g}$ valued 1 form satisfying relations (49a) and (51a,c) with j, l, ω substituted by $j_{\mathrm{dR}}, l_{\mathrm{dR}}, y$, respectively [33]. We denote by $\mathrm{Conn}(M)$ the affine space of the connections on M.

Theorem 4. *One has*

$$\mathrm{Conn}(\mathsf{f}) \simeq \mathrm{Conn}(M) \tag{125}$$

(cf. Definition 9).

Proof. Any $a \in \mathsf{f}^1 \otimes \mathfrak{g}$ is locally of the form $a = a_i \tilde{x}^i$, where a_i is a $\mathfrak{g}$ valued smooth map. Define $\lambda(a) = a_i d_{\mathrm{dR}} x^i$. Then, by the above remarks, $\lambda(a)$ is a connection of M if and only if a is a connection of f. The map λ is obviously a bijection. □

8.2. $N = 2$ *differential geometry*

Definition 12. An $N = 2$ differential structure on M is a collection $\{(U_a, X_a)|a \in A\}$, where
1. $\{U_a|a \in A\}$ is an open covering of M;
2. for each $a \in A$, $X_a : U_a \mapsto (\mathsf{S}^0_2)^m$ and $x_a = X_a|_{\theta=0} : U_a \mapsto \mathbb{R}^m$ is a coordinate of M;
3. for $a, b \in A$ such that $U_a \cap U_b \neq \emptyset$, $X_a = x_a \circ x_b^{-1}(X_b)$.

Below, we shall omit the chart indices $a, b, \ldots$ except when dealing with matching relations.

We write as usual

$$X^i = x^i + \theta^A x^i{}_{,A} + \tfrac{1}{2}\epsilon_{KL}\theta^K\theta^L\tilde{x}^i, \quad X^i{}_{,A} = x^i{}_{,A} + \epsilon_{AK}\theta^K\tilde{x}^i, \quad \tilde{X}^i = \tilde{x}^i, \tag{128a–c}$$

where $x^i : U \mapsto \mathbb{R}$, $x^i{}_{,A} : U \mapsto \mathsf{s}^1$, $\tilde{x}^i : U \mapsto \mathsf{s}^2$.

We introduce the $N = 2$ covariant superderivatives

$$D_i = \tilde{\partial}_{xi} + \epsilon_{KL}\theta^K\partial^{,L}_{xi} + \tfrac{1}{2}\epsilon_{KL}\theta^K\theta^L\partial_{xi}, \quad D_{i,A} = \epsilon_{AK}(\partial^{,K}_{xi} + \theta^K\partial_{xi}), \quad \tilde{D}_i = \partial_{xi}, \tag{129a–c}$$

where $\partial_{xi}{}^{;A} = \partial/\partial x^i{}_{,A}$, $\tilde\partial_{xi} = \partial/\partial \tilde x^i$. One has

$$[D_i, D_j] = 0, \quad [D_i, D_{j,A}] = 0, \quad [D_i, \tilde D_j] = 0,$$
$$[D_{i,A}, D_{j,B}] = 0, \quad [D_{i,A}, \tilde D_j] = 0, \quad [\tilde D_i, \tilde D_j] = 0. \tag{130a–f}$$

Further,

$$D_i X^j = 0, \quad D_i X^j{}_{,A} = 0, \quad D_i \tilde X^j = \delta_i^j, \quad D_{i,A} X^j = 0, \quad D_{i,A} X^j{}_{,B} = \epsilon_{AB}\delta_i^j,$$
$$D_{i,A}\tilde X^j = 0, \quad \tilde D_i X^j = \delta_i^j, \quad \tilde D_i X^j{}_{,A} = 0, \quad \tilde D_i \tilde X^j = 0. \tag{131a–i}$$

By (128), relations (131) completely characterize D_i, $D_{i,A}$, $\tilde D_i$.

The transformation properties of X^i under chart changes, stated in Definition 12, imply that

$$X_a{}^i{}_{,A} = X_b{}^j{}_{,A}\tilde D_{bj} X_a{}^i, \quad \tilde X_a{}^i = \tilde X_b{}^j \tilde D_{bj} X_a{}^i + \tfrac12 \epsilon^{JK} X_b{}^j{}_{,J} X_b{}^k{}_{,K}\tilde D_{bj}\tilde D_{bk} X_a{}^i. \tag{132a,b}$$

Using that (131) completely characterize D_i, $D_{i,A}$, $\tilde D_i$, one can show easily that they match as

$$D_{ai} = \tilde D_{ai} X_b{}^j D_{bj}, \quad D_{ai,A} = \tilde D_{ai} X_b{}^j{}_{,A} D_{bj} + \tilde D_{ai} X_b{}^j D_{bj,A},$$
$$\tilde D_{ai} = \tilde D_{ai}\tilde X_b{}^j D_{bj} + \epsilon^{KL}\tilde D_{ai} X_b{}^k{}_{,K} D_{bk,L} + \tilde D_{ai} X_b{}^j \tilde D_{bj}. \tag{133a–c}$$

We denote by $\mathcal F$ the sheaf of germs of smooth $N = 2$ functions on M generated by X^i, $X^i{}_{,A}$, $\tilde X^i$. By definition, a generic element $F \in \mathcal F(U)$ is a finite sum of the form $F = \sum_{p,q\ge0} f^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} \circ X X^{i_1}{}_{,I_1}\cdots X^{i_p}{}_{,I_p}\tilde X^{i_{p+1}}\cdots \tilde X^{i_{p+q}}$ for certain smooth maps $f^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} : \mathbb R^m \mapsto \mathbb R$ antisymmetric in the pairs $(i_1, I_1), \dots, (i_p, I_p)$ and symmetric in $i_{p+1}, \dots, i_{p+q}$. It is straightforward though tedious to show that

$$\begin{aligned} F &= \sum_{p=0}^{2m}\sum_{q=0}^{q_0}\{F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_1}{}_{,I_1} x^{i_2}{}_{,I_2} \\ &+\theta^K[\delta_K^{I_0}\partial_{i_0}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_0}{}_{,I_0}x^{i_1}{}_{,I_1}x^{i_2}{}_{,I_2} - p\epsilon_{KI_1}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_2}{}_{,I_2}\tilde x^{i_1}] \\ &+\tfrac12\epsilon_{KL}\theta^K\theta^L[\tfrac12\epsilon^{I_{-1}I_0}\partial_{i_{-1}}\partial_{i_0}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_{-1}}{}_{,I_{-1}}x^{i_0}{}_{,I_0}x^{i_1}{}_{,I_1}x^{i_2}{}_{,I_2} \\ &+\partial_{i_0}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_1}{}_{,I_1}x^{i_2}{}_{,I_2}\tilde x^{i_0} - p\delta^{I_0}_{I_1}\partial_{i_0}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} x^{i_0}{}_{,I_0}x^{i_2}{}_{,I_2}\tilde x^{i_1} \\ &+\tfrac12 p(p-1)\epsilon_{I_1I_2}F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}}\tilde x^{i_1}\tilde x^{i_2}]\}x^{i_3}{}_{,I_3}\cdots x^{i_p}{}_{,I_p}\tilde x^{i_{p+1}}\cdots\tilde x^{i_{p+q}}, \end{aligned} \tag{134}$$

where $F^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} = f^{I_1\cdots I_p}_{i_1\cdots i_p i_{p+1}\cdots i_{p+q}} \circ x$. Notice that F is completely determined by $f = F|_{\theta=0}$.

$\mathcal F$ has a natural grading corresponding to the total s degree of $x^i{}_{,I}$, $\tilde x^i$.

We define on $U_a \cap U_b \neq \emptyset$,

$$Z_{ab}{}^i{}_j = \tilde D_{bj} X^i_a. \tag{135}$$

It is easy to see that Z is a $\mathrm{GL}(m, \mathcal{F})$ 1-cocycle on M. Z is called the fundamental 1 cocycle of the $N = 2$ differential structure. One can introduce in standard fashion the sheaf $\mathcal{F}_{r,s} := \mathcal{F}(Z^{\otimes r} \otimes Z^{\vee\otimes s})$ of germs of smooth $N = 2$ sections of $Z^{\otimes r} \otimes Z^{\vee\otimes s}$. We denote by $\mathsf{f}_{r,s}$ the vector space of sections of $\mathcal{F}_{r,s}$ on M.

$z = Z|_{\theta=0}$ is nothing but the tangent bundle 1-cocycle of M. However, unlike the $N = 1$ case, there is no simple geometrical interpretation of the spaces $\mathsf{f}^p_{r,s}$.

We are particularly interested in the space $\mathsf{f}_{0,0}$, which is a graded algebra.

We define

$$H_A = -X^i{}_{,A} D_i, \quad H_{A,B} = X^i{}_{,A} D_{i,B} - \epsilon_{AB}\tilde{X}^i D_i \quad \tilde{H}_A = \tilde{X}^i D_{i,A} - s X^i{}_{,A} \tilde{D}_i. \tag{136a–c}$$

Using (132) and (133), it is easy to see that H_A, $H_{A,B}$, $\tilde{H}_A$ are globally defined derivations on $\mathsf{f}_{0,0}$.

We set next

$$\begin{aligned} &I(\xi) = C^i\xi D_i, \quad I_{,A}(\xi) = X^j{}_{,A}\tilde{D}_j C^i\xi D_i + C^i\xi D_{i,A}, \\ &\tilde{I}(\xi) = [\tilde{X}^j\tilde{D}_j C^i\xi + \tfrac{1}{2}\epsilon^{KL}X^k{}_{,K}X^l{}_{,L}\tilde{D}_k\tilde{D}_l C^i\xi]D_i + \epsilon^{KL}X^k{}_{,K}\tilde{D}_k C^i\xi D_{i,L} + C^i\xi\tilde{D}_i, \end{aligned} \tag{137a–c}$$

where $C\xi$ is the element of $\mathsf{f}^0_{1,0}$ corresponding to $c\xi$ and is given explicitly by $C^i\xi = c^i\xi + \theta^K x^j{}_{,K}\partial_{xj}c^i\xi + \frac{1}{2}\epsilon_{KL}\theta^K\theta^L[\tilde{x}^j\partial_{xj}c^i\xi + \frac{1}{2}\epsilon^{MN}x^j{}_{,M}x^k{}_{,N}\partial_{xj}\partial_{xk}c^i\xi]$. By (133a) and (135), $I(\xi)$, $I_{,A}(\xi)$, $\tilde{I}(\xi)$ are globally defined derivations on $\mathsf{f}_{0,0}$.

Using the relation $D_i C^j\xi = 0$, $D_{i,A}C^j\xi = 0$, it is now straightforward to verify that H_A, $H_{A,B}$, $\tilde{H}_A$, I, $I_{,A}$, $\tilde{I}$ satisfy relations (35)–(37). In this way, $\mathsf{f}_{0,0}$ becomes a $\mathbb{Z}$ graded left module algebra of the $\mathbb{Z}$ graded Lie algebra t (cf. Section 3.2).

Thus, $\mathsf{f} := \mathsf{f}_{0,0}$ acquires the structure of $N = 2\,\mathfrak{g}$ superoperation (cf. Definition 8), the relevant graded derivations being

$$h_A = -x^i{}_{,A}\tilde{\partial}_{xi}, \quad h_{A,B} = x^i{}_{,A}\epsilon_{BL}\partial_{xi}^{;L} - \epsilon_{AB}\tilde{x}^i\tilde{\partial}_{xi}, \quad \tilde{h}_A = \tilde{x}^i\epsilon_{AL}\partial_{xi}^{;L} - x^i{}_{,A}\partial_{xi}, \tag{138a–c}$$

$$\begin{aligned} &i(\xi) = c^i\xi\tilde{\partial}_{xi}, \quad i_{,A}(\xi) = c^i\xi\epsilon_{AL}\partial_{xi}^{;L} + x^j{}_{,A}\partial_{xj}c^i\xi\tilde{\partial}_{xi}, \\ &\tilde{i}(\xi) = c^i\xi\partial_{xi} + x^j{}_{,K}\partial_{xj}c^i\xi\partial_{xi}^{;K} + [\tilde{x}^j\partial_{xj}c^i\xi + \tfrac{1}{2}\epsilon^{KL}x^k{}_{,K}x^l{}_{,L}\partial_{xk}\partial_{xl}c^i\xi]\tilde{\partial}_{xi}. \end{aligned} \tag{139a–c}$$

This superoperation is canonically associated to the $N = 2$ differential structure.

In spite of the fact that, in the $N = 2$ case, f does not have any simple geometrical interpretation, unlike its $N = 1$ counterpart, the (basic) cohomology of f in the $N = 2$ case has essentially the same content as that of the $N = 1$ case and a theorem analogous to Theorem 3 holds.

Theorem 5. *There is an isomorphism of the $N = 2$ (basic) cohomology of* f *the de Rham (basic) cohomology of the (G) manifold M. Indeed, one has that $H^{n,p}(\mathsf{f}) = 0(H^{n,p}_{\mathrm{basic}}(\mathsf{f}) = 0)$, except perhaps for $(n, p) = (1, 0)$, $(r, r + 1)$ with $1 \le r \le m$, and*

$$H^{1,0}(\mathsf{f}) \simeq H^0_{\mathrm{dR}}(M), \quad H^{r,r+1}(\mathsf{f}) \simeq H^r_{\mathrm{dR}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2, \quad 1 \le r \le m, \tag{140a,b}$$

$$H^{1,0}_{\mathrm{basic}}(\mathsf{f}) \simeq H^0_{\mathrm{dR\ \ basic}}(M), \quad H^{r,r+1}_{\mathrm{basic}}(\mathsf{f}) \simeq H^r_{\mathrm{dR\ \ basic}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2, \quad 1 \le r \le m. \tag{141a,b}$$

Proof. By Proposition 4, $H^{n,p}(\mathsf{f}) = 0\,(H^{n,p}_{\mathrm{basic}}(\mathsf{f}) = 0)$ except perhaps for $p = \pm n + 1$. On the other hand, from the definition of f, given above, $\mathsf{f}^{n,p} = 0$ for $p < 0$. So, $H^{n,p}(\mathsf{f}) = 0\,(H^{n,p}_{\mathrm{basic}}(\mathsf{f}) = 0)$ except perhaps for $(n, p) = (1, 0)$, $(r, r + 1)$ with $1 \le r$. Consider first the case where $(n, p) = (1, 0)$. From (136b) and the representation theory of $\mathsf{i} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, it is immediate to see that $\mathsf{f}^{1,0}$ consists precisely of the F of the form $F = \alpha$ for some smooth function α on M and that $\mathsf{f}^{1,-2} = 0$. Further, the conditions $d_A F = 0$ is equivalent to $d_{\mathrm{dR}}\alpha = 0$, hence to the local constance of α. We thus have a linear bijection $\nu : \mathsf{f}^{1,0} \cap \cap_{A=1,2} \ker d_A \mapsto Z^0_{\mathrm{dR}}(M)$, where $Z^r_{\mathrm{dR}}(M)$ is the space of closed r forms, given by $F \mapsto \alpha$. Being $\mathsf{f}^{1,-2} = 0$, (140a) follows. (141a) also holds, as, clearly, $\mathsf{f}^{1,0} = \mathsf{f}^{1,0}_{\mathrm{basic}}$ and $Z^0_{\mathrm{dR}}(M) = Z^0_{\mathrm{dR\ \ basic}}(M)$. Consider next the case where $(n, p) = (r, r + 1)$ with $1 \le r$. Let $F \in \mathsf{f}^{r,r+1}$. From (136b) and the representation theory of $\mathsf{i} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, F is locally of the form

$$F = x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} [\tilde{x}^{i_r} \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r} + \tfrac{1}{2} \epsilon^{MN} x^{i_r}_{,M} x^{i_{r+1}}_{,N} \beta^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r i_{r+1}}] \tag{142}$$

with $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r}$ a smooth map symmetric in $A_1, \ldots, A_{r-1}$ and antisymmetric in $i_1, \ldots, i_{r-1}$ and $\beta^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r i_{r+1}}$ a smooth map symmetric in $A_1, \ldots, A_{r-1}$, antisymmetric in $i_1, \ldots, i_{r-1}$ and symmetric in i_r, i_{r+1}. Next, assume that $d_A F = 0$. Substituting (142) into the relation $d_A F = 0$ and taking into account the fact that terms with different numbers of $x^i_{,I}, \tilde{x}^i$ are linearly independent and, thus, must vanish separately, one gets the following three identities

$$x^{i_1}_{,A_1} \cdots x^{i_{r-2}}_{,A_{r-2}} \tilde{x}^{i_{r-1}} \tilde{x}^{i_r} \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r} = 0, \tag{143}$$

$$\begin{aligned} &(r-1)\epsilon_{AA_{r-1}} x^{i_1}_{,A_1} \cdots x^{i_{r-2}}_{,A_{r-2}} \tfrac{1}{2} \epsilon^{MN} x^{i_r}_{,M} x^{i_{r+1}}_{,N} \tilde{x}^{i_{r-1}} \beta^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r i_{r+1}} \\ &\quad + x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} [-x^{i_r}_{,A} \tilde{x}^{i_{r+1}} \beta^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r i_{r+1}} \quad + x^{i_{r+1}}_{,A} \tilde{x}^{i_r} \partial_{x i_{r+1}} \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r}] = 0, \end{aligned} \tag{144}$$

$$x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} \tfrac{1}{2} \epsilon^{MN} x^{i_r}_{,M} x^{i_{r+1}}_{,N} x^{i_{r+2}}_{,A} \partial_{x i_{r+2}} \beta^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r i_{r+1}} = 0. \tag{145}$$

From (143), using the symmetry properties of $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r}$ and the fact that $x^i_{,A}, \tilde{x}^i$ are odd, even, respectively, it follows immediately that $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-2} i_{r-1} i_r} + \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-2} i_r i_{r-1}} = 0$. Since $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r}$ is already antisymmetric in $i_1, \ldots, i_{r-1}$, $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_r}$ is antisymmetric in all the indices $i_1, \ldots, i_r$. Thus, for fixed $A_1, \ldots, A_{r-1}$, the $\alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_r}$ are the coefficients of a local r form $\alpha^{A_1\cdots A_{r-1}}$. Next, applying the derivation u_B (cf. Eq. (138a)) to Eq. (144) and contracting with ϵ^{BA}, one gets

$$x^{i_1}_{,A_1}\cdots x^{i_{r-1}}_{,A_{r-1}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}_{,M}x^{i_{r+1}}_{,N}\beta^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_ri_{r+1}} = \frac{2}{r+1}x^{i_1}_{,A_1}\cdots x^{i_{r-1}}_{,A_{r-1}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}_{,M}x^{i_{r+1}}_{,N}\partial_{xi_{r+1}}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}. \tag{146}$$

Applying d_A to this relation, one gets

$$\begin{aligned}&(r-1)\epsilon_{AA_{r-1}}x^{i_1}{}_{,A_1}\cdots x^{i_{r-2}}{}_{,A_{r-2}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}{}_{,M}x^{i_{r+1}}{}_{,N}\tilde{x}^{i_{r-1}}\beta^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_ri_{r+1}}\\&\quad -x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}x^{i_r}{}_{,A}\tilde{x}^{i_{r+1}}\beta^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_ri_{r+1}}\\&\quad =2\frac{r-1}{r+1}\epsilon_{AA_{r-1}}x^{i_1}{}_{,A_1}\cdots x^{i_{r-2}}{}_{,A_{r-2}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}{}_{,M}x^{i_{r+1}}{}_{,N}\tilde{x}^{i_{r-1}}\partial_{xi_{r+1}}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}\\&\quad -\frac{1}{r+1}x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}x^{i_r}{}_{,A}\tilde{x}^{i_{r+1}}(\partial_{xi_{r+1}}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}+\partial_{xi_r}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_{r+1}}),\end{aligned} \tag{147}$$

$$x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}{}_{,M}x^{i_{r+1}}{}_{,N}x^{i_{r+2}}{}_{,A}\partial_{xi_{r+2}}\beta^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_ri_{r+1}} = \frac{2}{r+1}x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}{}_{,M}x^{i_{r+1}}{}_{,N}x^{i_{r+2}}{}_{,A}\partial_{xi_{r+1}}\partial_{xi_{r+2}}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}. \tag{148}$$

Substituting (147) and (148) into (144) and (145), respectively, one obtains after a straightforward calculation the equations

$$x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}x^{i_{r+1}}{}_{,A_{r+1}}\tilde{x}^{i_r}\sum_{l=1}^{r+1}(-1)^{l-1}\partial_{xi_l}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{l-1}i_{l+1}\cdots i_{r+1}}=0, \tag{149}$$

$$x^{i_1}{}_{,A_1}\cdots x^{i_{r-1}}{}_{,A_{r-1}}\tfrac{1}{2}\epsilon^{MN}x^{i_r}{}_{,M}x^{i_{r+1}}{}_{,N}x^{i_{r+2}}{}_{,A}\partial_{xi_{r+1}}\partial_{xi_{r+2}}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}=0. \tag{150}$$

Using the symmetry properties of $\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}i_r}$ and the fact that $x^i{}_{,A}$, $\tilde{x}^i$ are odd, even, respectively, it is easy to see that (149) implies that $\sum_{l=1}^{r+1}(-1)^{l-1}\partial_{xi_l}\alpha^{A_1\cdots A_{r-1}}_{i_1\cdots i_{l-1}i_{l+1}\cdots i_{r+1}}=0$ or $d_{\mathrm{dR}}\alpha^{A_1\cdots A_{r-1}}=0$ so that the local r form $\alpha^{A_1\cdots A_{r-1}}$ is closed and locally exact. By this reason and the fact that $x^i{}_{,I}\partial_{xi}x^j{}_{,J}\partial_{xj}x^k{}_{,K}\partial_{xk}=0$ by antisymmetry, one finds that Eq. (150) is automatically satisfied. We note that, by (132a) and the global definition of F, it is easy to see the local exact r form $\alpha^{A_1\cdots A_{r-1}}$ is the local restriction of a globally defined closed r form, which will be denoted by the same symbol. To summarize, we have shown that (143)–(145) imply that, for fixed $A_1,\ldots,A_{r-1}$, $\alpha^{A_1\cdots A_{r-1}}$ is a closed r form and that (146) holds. Conversely, assume that for fixed $A_1,\ldots,A_{r-1}$, $\alpha^{A_1\cdots A_{r-1}}$ is a closed r form and that (146) holds. Using (132a,b), it is straightforward though tedious to show that F, as given by (142), belongs to $\mathfrak{f}^{r,r+1}$. As shown above, (146) implies (147) and (148) using which Eqs. (144) and (145) become equivalent to Eqs. (149) and (150). Eqs. (143), (149) and (150), are trivially satisfied by the closed r form $\alpha^{A_1\cdots A_{r-1}}$. Thus, (143)–(145) are satisfied as well implying that $d_A F = 0$. In conclusion, we have shown that $\mathfrak{f}^{r,r+1} \cap \cap_{A=1,2} \ker d_A$ consists precisely of the elements $F \in \mathfrak{f}^{r,r+1}$ of the form

$$F = x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} \left[\tilde{x}^{i_r} \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r} + \frac{2}{r+1} \frac{1}{2} \epsilon^{MN} x^{i_r}_{,M} x^{i_{r+1}}_{,N} \partial_{x i_{r+1}} \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r} \right] \tag{151}$$

with $\alpha^{A_1 \cdots A_{r-1}}$ an r form symmetric in $A_1, \ldots, A_{r-1}$ and such that $d_{\mathrm{dR}} \alpha^{A_1 \cdots A_{r-1}} = 0$. We thus have a linear bijection $\nu : \mathfrak{f}^{r,r+1} \cap \cap_{A=1,2} \ker d_A \mapsto Z^r_{\mathrm{dR}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2$, where $Z^r_{\mathrm{dR}}(M)$ is the space of closed r forms, given by $F \mapsto (\alpha^{A_1 \cdots A_{r-1}})_{A_1, \ldots, A_{r-1}=1,2}$. Next, assume that $F \in \frac{1}{2} \epsilon^{KL} d_K d_L \mathfrak{f}^{r,r-1}$. Then, $F = \frac{1}{2} \epsilon^{KL} d_K d_L G$ for some $G \in \mathfrak{f}^{r,r-1}$. From (136b) and the representation theory of $\mathfrak{i} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathbb{R}$, G is of the form

$$G = x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} \gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1}}, \tag{152}$$

with $\gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1}}$ a smooth map symmetric in $A_1, \ldots, A_{r-1}$ and antisymmetric in $i_1, \ldots, i_{r-1}$. By a straightforward computation, one finds that

$$\begin{aligned} \frac{1}{2} \epsilon^{KL} d_K d_L G = (-1)^{r-1} x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} \Bigg[& \tilde{x}^{i_r} \sum_{l=1}^{r} (-1)^{l-1} \partial_{x i_l} \gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r-1} i_r} \\ & + \frac{2}{r+1} \frac{1}{2} \epsilon^{MN} x^{i_r}_{,M} x^{i_{r+1}}_{,N} \partial_{x i_{r+1}} \sum_{l=1}^{r} (-1)^{l-1} \partial_{x i_l} \gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{l-1} i_{l+1} \cdots i_{r-1} i_r} \Bigg]. \end{aligned} \tag{153}$$

Note that $\gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1}}$ are the coefficients of a local $r-1$ form $\gamma^{A_1 \cdots A_{r-1}}$. By (132a) and the global definition of G, $\gamma^{A_1 \cdots A_{r-1}}$ is the restriction of a globally defined $r-1$ form, which we shall denote by the same symbol. As (153) indicates, the linear map ν maps cohomologically trivial elements of $\mathfrak{f}^{r,r+1} \cap \cap_{A=1,2} \ker d_A$ into cohomologically trivial elements of $Z^r_{\mathrm{dR}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2$. Thus, ν induces a linear bijection $\hat{\nu} : H^{r,r+1}(\mathfrak{f}) \mapsto H^r_{\mathrm{dR}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2$. Next, assume that $F \in \mathfrak{f}^{r,r+1}_{\mathrm{basic}}$ and that $d_A F = 0$. In particular, F is of the form (151) for some closed r form $\alpha^{A_1 \cdots A_{r-1}}$ symmetric in $A_1, \ldots, A_{r-1}$. By (43d,e) and the relation $d_A F = 0$, the basicity of F is equivalent to the relation $j(\xi) F = 0, \xi \in \mathfrak{g}$, where $j(\xi)$, by (39a), is given in the present situation by (139a). A simple computation shows that this identity is equivalent to

$$x^{i_1}_{,A_1} \cdots x^{i_{r-1}}_{,A_{r-1}} c^{i_r} \xi \alpha^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1} i_r} = 0. \tag{154}$$

As is straightforward to check, this relation entails that $c^{i_0} \xi \alpha^{A_1 \cdots A_{r-1}}_{i_0 i_1 \cdots i_{r-1}} = 0$, so that $j_{\mathrm{dR}}(\xi) \alpha^{A_1 \cdots A_{r-1}} = 0$. As $l_{\mathrm{dR}}(\xi) = [d_{\mathrm{dR}}, j_{\mathrm{dR}}(\xi)]$ and $d_{\mathrm{dR}} \alpha^{A_1 \cdots A_{r-1}} = 0$, the closed r form $\alpha^{A_1 \cdots A_{r-1}}$ is basic. Conversely, if $\alpha^{A_1 \cdots A_{r-1}}$ is basic (154) obviously holds. So, the linear bijection ν introduced earlier maps $\mathfrak{f}^{r,r+1}_{\mathrm{basic}} \cap \cap_{A=1,2} \ker d_A$ into $Z^r_{\mathrm{dR\ basic}}(M) \otimes \bigvee^{r-1} \mathbb{R}^2$, where $Z^r_{\mathrm{dR\ basic}}(M)$ is the space of closed basic r forms. Let $G \in \mathfrak{f}^{r,r-1}_{\mathrm{basic}}$. Then, G is of the form (152) and satisfies $j(\xi) G = 0$, $j_A(\xi) G = 0$, $l(\xi) G = 0$, where $j(\xi)$, $j_A(\xi)$ and $l(\xi)$ are defined by (39) and are given by (139). It is straightforward to see that these identities yield the equations

$$\epsilon_{A A_{r-1}} x^{i_1}_{,A_1} \cdots x^{i_{r-2}}_{,A_{r-2}} c^{i_{r-1}} \xi \gamma^{A_1 \cdots A_{r-1}}_{i_1 \cdots i_{r-1}} = 0, \tag{155}$$

$$x^{i_1}_{,A_1}\cdots x^{i_{r-1}}_{,A_{r-1}}\left[\sum_{l=1}^{r-1}\partial_{xi_l}c^{i_r}\xi\gamma^{A_1\cdots A_{r-1}}_{i_1\cdots i_{l-1}i_r i_{l+1}\cdots i_{r-1}}+c^{i_r}\xi\partial_{xi_r}\gamma^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}}\right]=0. \tag{156}$$

Thus, $c^{i_0}\xi\gamma^{A_1\cdots A_{r-1}}_{i_0 i_1\cdots i_{r-2}}=0$, $\sum_{l=1}^{r-1}\partial_{xi_l}c^{i_r}\xi\gamma^{A_1\cdots A_{r-1}}_{i_1\cdots i_{l-1}i_r i_{l+1}\cdots i_{r-1}}+c^{i_r}\xi\partial_{xi_r}\gamma^{A_1\cdots A_{r-1}}_{i_1\cdots i_{r-1}}=0$, as is easy to see, so that $j_{\mathrm{dR}}(\xi)\gamma^{A_1\cdots A_{r-1}}=0$ and $l_{\mathrm{dR}}(\xi)\gamma^{A_1\cdots A_{r-1}}=0$ and $\gamma^{A_1\cdots A_{r-1}}$ is basic. Conversely the basicity of $\gamma^{A_1\cdots A_{r-1}}$ implies (155) and (156). From (152) and (153), we see that ν maps cohomologically trivial elements of $\mathfrak{f}^{r,r+1}_{\mathrm{basic}}\cap\cap_{A=1,2}\ker d_A$ into cohomologically trivial elements of $Z^r_{\mathrm{dR\ \ basic}}(M)\otimes\bigvee^{r-1}\mathbb{R}^2$. Thus, ν induces a linear bijection $\hat\nu: H^{r,r+1}_{\mathrm{basic}}(\mathfrak{f})\mapsto H^r_{\mathrm{dR\ \ basic}}(M)\otimes\bigvee^{r-1}\mathbb{R}^2$. □

A theorem analogous to Theorem 4 also holds.

Theorem 6. *One has*

$$\mathrm{Conn}(\mathfrak{f})\simeq\mathrm{Conn}(M) \tag{157}$$

(cf. Definition 10*).*

Proof. From the representation theory of $\mathfrak{i}=\mathfrak{sl}(2,\mathbb{R})\oplus\mathbb{R}$, any $a_A\in\mathfrak{f}^{2,1}\otimes\mathfrak{g}$ is locally of the form $a_A=a_i\tilde{x}^i_{,A}$, where a_i is a $\mathfrak{g}$ valued smooth map. Define $\lambda((a_A)_{A=1,2})=a_i d_{\mathrm{dR}}x^i$. Then, from (139), it is easy to see that $\lambda((a_A)_{A=1,2})$ is a connection of M if and only if $(a_A)_{A=1,2}$ is a connection of $\mathfrak{f}$. The map λ is clearly a bijection. □

8.3. The relation between the $N=1$ and $N=2$ cohomologies of $\mathfrak{f}$

Let $\mathfrak{f}(n)$ denote the superoperation $\mathfrak{f}$ for $N=n,\ n=1,2$, as defined in Sections 8.1 and 8.2.

Corollary 2. *One has*

$$H^{n,\pm n+1}(\mathfrak{f}(2))\simeq H^{\pm(n-1/2)+1/2}(\mathfrak{f}(1))\otimes\bigvee^{n-1}\mathbb{R}^2, \tag{158}$$

$$H^{n,\pm n+1}_{\mathrm{basic}}(\mathfrak{f}(2))\simeq H^{\pm(n-1/2)+1/2}_{\mathrm{basic}}(\mathfrak{f}(1))\otimes\bigvee^{n-1}\mathbb{R}^2. \tag{159}$$

Proof. Combine Theorem 3 and Theorem 5. □

Thus, the $N=1$ and $N=2$ cohomologies of $\mathfrak{f}$ are closely related. Note the analogy to relations (85) and (86).

Corollary 3. *One has*

$$\mathrm{Conn}(\mathfrak{f}(2))\simeq\mathrm{Conn}(\mathfrak{f}(1)). \tag{160}$$

Proof. Combine Theorem 4 and Theorem 6. □

Thus, the $N = 1$ and $N = 2$ connections of $\mathfrak{f}$ are manifestations of the same geometrical structure.

9. Concluding remarks

There are a few fundamental questions which are still open and which are of considerable salience both in geometry and topological field theory.

Corollaries 1 suggest that a relation formally analogous to (159) should hold also between the $N = 1$ and $N = 2$ equivariant cohomologies of $\mathfrak{f}$ (cf. Section 7). Further, from (160), we expect that the range of the $N = 1$ and $N = 2$ Weil homomorphisms (cf. Sections 7.1 and 7.2) should have essentially the same content. This question is of fundamental importance to show conclusively that balanced topological gauge field theory does not contain new topological observables besides those coming from the underlying $N = 1$ theory. We have not been able to either prove or disprove such assertions yet.

There are other possible lines of inquiry. It is known that the $N = 1$ Maurer–Cartan equations of a Lie algebra $\mathfrak{g}$ can be obtained from the $N = 1$ Weil algebra relation (50) by formally setting $\phi = 0$. By a similar procedure, one can obtain the $N = 2$ Maurer–Cartan equations by formally setting $\phi_{AB} = 0$, $\rho_A = 0$ in the $N = 2$ Weil algebra relations (56). Indeed, it is straightforward to check that the basic relation $[d_A, d_B] = 0$ still holds after this truncation. This hints to a possible $N = 2$ generalization of gauge fixing.

Finally, note that, by obtaining the $N = 2$ Weil algebra, we are in the position of formulating other models of equivariant cohomology in balanced topological field theory besides Cartan's used in [31], generalizing the $N = 1$ intermediate or BRST model of [7,8].

We leave these matters to future work.

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